

Technical Appendix to ANALYSIS OF ACTIVE PORTFOLIO MANAGEMENT

For those acquainted with matrix algebra, more complete descriptions of the fundamental law parameters are based on an N -by-1 vector of forecasted active returns for the assets, $\boldsymbol{\mu}$, and an N -by- N matrix of estimated active return covariances, $\boldsymbol{\Omega}$, also called the “risk model.” Note that both $\boldsymbol{\mu}$ and $\boldsymbol{\Omega}$ are for active returns, which can be calculated from their total return counterparts.¹ The objective function of active portfolio management is to choose the N -by-1 vector of active weights for the assets, \mathbf{w} , to maximize the expected active portfolio return,

$$E(R_A) = \boldsymbol{\mu}'\mathbf{w} \quad (\text{A1})$$

subject to a limit on active risk,

$$\sigma_A = \sqrt{\mathbf{w}'\boldsymbol{\Omega}\mathbf{w}} \quad (\text{A2})$$

The well-known solution to this optimization problem is

$$\mathbf{w}^* = \frac{\sigma_A}{\sqrt{\boldsymbol{\mu}'\boldsymbol{\Omega}^{-1}\boldsymbol{\mu}}} \boldsymbol{\Omega}^{-1}\boldsymbol{\mu} \quad (\text{A3})$$

where the * designates optimal asset active weights. We assume that the active returns for the assets are scaled using a matrix version of the Grinold (1994) rule:

$$\boldsymbol{\mu} = (\text{IC})\boldsymbol{\Lambda}\mathbf{S} \quad (\text{A4})$$

where IC is the investor-specified expected information coefficient, \mathbf{S} is an N -by-1 vector of asset scores, and $\boldsymbol{\Lambda}$ is a square matrix of the assets’ benchmark residual risks. Specifically, $\boldsymbol{\Lambda}$

¹ The forecasted total returns for the assets, $\boldsymbol{\mu}_T$, can be adjusted for a common benchmark return and shifted to ensure that the subsequent budget constraint is met by the single formula $\boldsymbol{\mu} = \boldsymbol{\mu}_T - \mu_0\mathbf{1}$, where the shift parameter is $\mu_0 = \boldsymbol{\mu}_T'\boldsymbol{\Omega}^{-1}\mathbf{1} / \mathbf{1}'\boldsymbol{\Omega}^{-1}\mathbf{1}$ ($\mathbf{1}$ is a vector of 1s). Similarly, the estimated active return risk model is based on the total return risk model, $\boldsymbol{\Omega} = \boldsymbol{\Omega}_T - \sigma_B^2\boldsymbol{\beta}_B\boldsymbol{\beta}_B'$, where $\sigma_B^2 = \mathbf{w}_B'\boldsymbol{\Omega}_T\mathbf{w}_B$, $\boldsymbol{\beta}_B = (1/\sigma_B^2)\boldsymbol{\Omega}_T\mathbf{w}_B$, and \mathbf{w}_B is an N -by-1 vector of the assets’ weights in the benchmark portfolio. Note that the total asset return covariance matrix, $\boldsymbol{\Omega}_T$, must be at least one asset larger than N to insure that $\boldsymbol{\Omega}$ is invertible. For example, the matrix can be formed from a general multi-factor risk model, $\boldsymbol{\Omega}_T = \mathbf{B}'\mathbf{V}\mathbf{B} + \boldsymbol{\Delta}$, where \mathbf{V} is a K -by- K factor return covariance matrix, \mathbf{B} is an at least $(N + 1)$ -by- K factor exposure matrix, $\boldsymbol{\Delta}$ is a diagonal idiosyncratic return matrix, and the vector \mathbf{w}_B has at least one extra non-zero weight. In subsequent calculations, the extra non-zero weights must be accounted for to calculate the true benchmark variance, σ_B^2 .

is constructed from the square root of the diagonal elements of $\mathbf{\Omega}$ placed along the diagonal and then filled with zero off-diagonal elements. In other words, the active return risk model is decomposed into volatilities and correlations by

$$\mathbf{\Omega} = \mathbf{\Lambda}\mathbf{\Pi}\mathbf{\Lambda} \quad (\text{A5})$$

where $\mathbf{\Pi}$ is the N -by- N active return correlation matrix. Substituting Equations A4 and A5 into A3 gives the optimal active weight vector as

$$\mathbf{w}^* = \frac{\sigma_A}{\sqrt{\mathbf{S}'\mathbf{\Pi}^{-1}\mathbf{S}}} \mathbf{\Pi}^{-1}\mathbf{S} \quad (\text{A6})$$

Substituting Equations A4 and A6 into the definition of the expected active portfolio return in Equation A1 gives one form of the fundamental law formula:

$$E(R_A)^* = (\text{IC})\sqrt{\mathbf{S}'\mathbf{\Pi}^{-1}\mathbf{S}}\sigma_A \quad (\text{A7})$$

However, in order to ensure that breadth is not a function of any specific set of scores, we define it as the sum of the elements in the inverse correlation matrix,

$$\text{BR} = \mathbf{1}'\mathbf{\Pi}^{-1}\mathbf{1} \quad (\text{A8})$$

and use a signal-adjusted information coefficient,

$$\text{IC}_{Adj} = (\text{IC})\sqrt{\frac{\mathbf{S}'\mathbf{\Pi}^{-1}\mathbf{S}}{\text{BR}}} \quad (\text{A9})$$

Substituting Equations A8 and A9 into A7 gives the basic fundamental law formula:

$$E(R_A)^* = (\text{IC}_{Adj})\sqrt{\text{BR}}\sigma_A \quad (\text{A10})$$

Let \mathbf{w} (without an $*$) be a vector of asset active weights from a numerical optimizer, with the objective to maximize the active portfolio return under constraints in addition to the active risk constraint, $\sigma_A = \sqrt{\mathbf{w}'\mathbf{\Omega}\mathbf{w}}$, and the standard budget constraint, $\mathbf{w}'\mathbf{1} = 0$. Using these constrained weights, the final fundamental law parameter is the transfer coefficient:

$$\text{TC} = \frac{\boldsymbol{\mu}'\mathbf{w}}{\sqrt{\boldsymbol{\mu}'\mathbf{\Omega}^{-1}\boldsymbol{\mu}}\sqrt{\mathbf{w}'\mathbf{\Omega}\mathbf{w}}} = \frac{E(R_A)}{E(R_A)^*} \quad (\text{A11})$$

where $E(R_A)$ is the expected active portfolio return using weights \mathbf{w} instead of \mathbf{w}^* . With the transfer coefficient in Equation A11, the full fundamental law formula is

$$E(R_A) = (\text{TC})(\text{IC}_{Adj})\sqrt{\text{BR}}\sigma_A \quad (\text{A12})$$

or in terms of the information ratio, $IR \equiv E(R_A) / \sigma_A$,

$$IR = (TC)(IC_{Adj})\sqrt{BR} \quad (\text{A13})$$

Equation A8, for breadth, uses the inverse correlation matrix, which has little interpretative value, but more intuitive calculations can be based on various simplifying assumptions for the risk model. For example, if the correlation matrix is diagonal (uncorrelated active returns), meaning that $\mathbf{\Pi}$ is the identity matrix, then the inverse correlation matrix, $\mathbf{\Pi}^{-1}$, is also the identity matrix; breadth is equal to the number of assets, $BR = N$; and the adjustment to IC in Equation A9 is not needed. A slightly more complex case is the constant correlation model of Elton and Gruber (1973), where all the off-diagonal elements of the correlation matrix are the same value, ρ . Under this assumption, it can be shown that breadth in Equation A8 is

$$BR = \frac{N}{1 + (N-1)\rho} \quad (\text{A14})$$

so that for positive values of ρ , breadth is lower than the number of assets. Similarly, under the constant correlation model, the adjustment to IC in Equation A9 is

$$IC_{Adj} = (IC)\sqrt{\frac{1 + (N-1)\rho}{1 - \rho}} \quad (\text{A15})$$

so that for positive values of ρ , the IC is adjusted upward. But again, if the active returns are uncorrelated ($\rho = 0$), then $BR = N$ in equation A14 and $IC_{Adj} = IC$ in Equation A15.