

READING

4

Performance Evaluation: Rate-of-Return Measurement

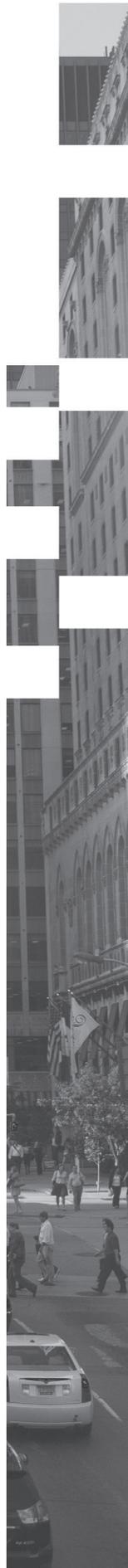
by Carl R. Bacon, CIPM, David R. Cariño, PhD, and Arin Stancil, CFA, CIPM

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LEARNING OUTCOMES

<i>Mastery</i>	<i>The candidate should be able to:</i>
<input type="checkbox"/>	a. explain the treatment of investment income (e.g., dividends and interest) in calculating holding period rates of return;
<input type="checkbox"/>	b. calculate and interpret the holding period rate of return on a stock or bond investment;
<input type="checkbox"/>	c. calculate and interpret a portfolio rate of return;
<input type="checkbox"/>	d. calculate the home currency equivalent of a non-domestic rate of return and explain a decomposition of the home currency equivalent return;
<input type="checkbox"/>	e. calculate and interpret rates of return that result from short positions and leveraged positions;
<input type="checkbox"/>	f. distinguish between book value and market value, realized and unrealized gains and losses, trade date accounting and settlement date accounting, and show how these elements are treated in determining the asset values to be used in calculating rates of return;
<input type="checkbox"/>	g. explain the major bases for calculating rates of return: nominal versus real, gross-of-fees versus net-of-fees, pre-tax versus post-tax, and leveraged versus cash basis;
<input type="checkbox"/>	h. explain and demonstrate discrete and continuous compounding;
<input type="checkbox"/>	i. convert rates of return to an annual basis and explain when annualization is or is not appropriate;
<input type="checkbox"/>	j. compare arithmetic and geometric mean rates of return;
<input type="checkbox"/>	k. explain arithmetic and geometric bases for calculating excess returns;
<input type="checkbox"/>	l. calculate and interpret money-weighted and time-weighted rates of return;
<input type="checkbox"/>	m. describe unit value pricing;

(continued)



LEARNING OUTCOMES

<i>Mastery</i>	<i>The candidate should be able to:</i>
<input type="checkbox"/>	n. explain the issues raised by external cash flows for performance evaluation;
<input type="checkbox"/>	o. explain major approximation methods to a true time-weighted rate of return;
<input type="checkbox"/>	p. compare the methods used to calculate composite returns;
<input type="checkbox"/>	q. explain the condition of consistency in calculating the time-weighted rates of return of portfolio segments and the overall portfolio.

1

INTRODUCTION

The **rate of return** on an investment over a specified time period is the percent gain or loss in wealth resulting from holding the investment over that period. The rate of return is an essential concept in investing. For example, in comparing investment managers, investors often evaluate the returns the managers have achieved over various time horizons and in various market conditions. When calculated properly, the rate of return enables valid comparisons to be made among managed portfolios, despite differences in the amounts invested and in other circumstances. Rates of return are measured and analyzed by performance analysts and investment consultants, who need to be proficient in calculating and interpreting them.

Rate of return is a *measurement*—a number calculated from more basic primitive data. The calculation of rates of return is the crucial first step in performance evaluation: Without accurate rates of return, we can make no further progress in analyzing performance. The purpose of this reading is to define the various forms of return useful for performance analysis, document methods of calculation, highlight calculation issues, provide guidance for appropriate use, and, hence, build the foundations for more detailed evaluation.

The reading is organized as follows: Section 2 presents the fundamentals of rate-of-return calculation over a single investment holding period without external cash flows (i.e., without client-initiated additions to or withdrawals from the portfolio). Section 3 covers the principles of compounding and averaging rates of return. Section 4 introduces rate-of-return measurement in the presence of external cash flows. Section 5 provides conclusions and a summary.

For those without an investment background or desiring a refresher, the Appendix to this reading provides a brief overview of the features of common shares and bonds and an explanation of the mathematical notations used in this reading.

2

SINGLE-PERIOD RATES OF RETURN

Performance analysts more commonly analyze portfolios than individual assets. An investment portfolio, or **portfolio** for short, is defined as a collection of one or more assets. Such assets may be common shares (common stock) and bonds, derivative

instruments, other securities, or even other portfolios.¹ A portfolio may consist of a single security, or it may comprise all the assets of a large pension fund. The principles of performance measurement apply equally to portfolios with small and large numbers of holdings and of any asset size.

The primary inputs to performance measurement are records of portfolio values over time. The holdings of a portfolio are the quantities (for example, numbers of shares in the case of stocks) of each asset in the portfolio. The value of the portfolio is the sum of the values of the component holdings; that sum is calculated by multiplying the quantities of the component holdings by their respective per unit values. Consider a portfolio made up of common stock and bonds. Five hundred shares of common stock valued at €30 per share give a value of €15,000 ($= 500 \times €30$) for the common stock component. Fifty €100 par value bonds with a market value of €98 each give a value of €4,900 ($= 50 \times €98$) for the bond component. The value of the portfolio is thus $€15,000 + €4,900 = €19,900$.

Specific procedures to determine value vary by asset type. For some assets, such as the shares of the largest publicly traded corporations, up-to-date market prices are readily available and are used for value. Where such market prices are not available, values might need to be estimated (a process called valuation or **appraisal**). Data for performance measurement are critically dependent on the accounting and record-keeping systems that maintain values of assets over time. Pricing sources, timing of records (e.g., concerning cash flows or corporate actions), and other variables may introduce differences in calculated performance.

2.1 Holding Period Rates of Return

A **holding period** is an interval between two points in time over which an asset or portfolio is assumed to be held. A holding period may be of any specified length and is typically based on calendar units, such as years, quarters, months, or days. The term **holding period rate of return** is used to underscore the time period over which the rate of return was earned and is defined as the percentage gain or loss in wealth resulting from holding some asset or portfolio over the stated holding period; thus it is calculated as the change in wealth resulting from holding some investment for a stated time period divided by the initial amount invested.

In this Section 2, on single-period rates of return, and the following Section 3, on multi-period rates of return, we assume that there are no deposits to or withdrawals from the portfolio, which are types of cash flows known as external cash flows. (Section 4 addresses calculating rates of return when external cash flows have taken place.)

Consider a single portfolio or security held over a given holding period. Given the beginning market value V_0 and the ending market value V_1 of the portfolio or security—and it is important to be clear to which V refers—the **return relative** (also referred to as the **value relative** or **wealth ratio**) is the ratio of ending (denoted “1”) to beginning (denoted “0”) value:

$$\text{Return relative} = \frac{V_1}{V_0}$$

In calculating return relatives, we need to take account of investment income that the owner of a portfolio or asset has received or is entitled to receive from holding the investment over the stated holding period. Thus, if V_1 represents the ending portfolio value, it must include income that the portfolio has received or is entitled to. If we are concerned with an individual security, a term for investment income that the owner of the security has received or is entitled to may need to accompany V_1 in the

¹ An example of a “portfolio of portfolios” is one that consists of shares of different US mutual funds or UK unit trusts (these are types of investments that pool the money of investors).

numerator. We will illustrate these points later. The holding period rate of return R is the return relative minus one, or, equivalently, the change in value over the holding period as a fraction of the beginning value:

$$R = \frac{V_1}{V_0} - 1 = \frac{V_1 - V_0}{V_0} \quad (1)$$

The result is usually expressed as a percentage (%). A few writers reserve “return” for a gain (loss) in units of money and “rate of return” for gains (losses) in relation to the amount invested, but most use the terms interchangeably in the sense of “rate of return.” Returns that are less than 1 percent are often expressed in **basis points**, abbreviated as **bps**, or hundredths of 1 percent. For example, if the beginning value of a portfolio is €100 and seven days later its value is €100.7, the seven-day holding period return is $(€100.7 - €100)/€100 = €0.7/€100 = 0.007$ or 0.7%, which can also be expressed as 70 bps (70/100 of 1 percent). Example 1 illustrates such a calculation.

EXAMPLE 1

Holding Period Rate-of-Return Calculations

- A six-month £10,000 certificate of deposit is redeemed at maturity for £10,200. The six-month holding period return is *closest* to:
 - 1.02%.
 - 1.96%.
 - 2.00%.
- A securities account had no deposit or withdrawal transactions during the month of April. Account values (in thousands) reported by the record keeper were as follows:

	31 March	30 April
Equity securities	€1,029,234.12	€1,108,314.67
Cash	€52,078.23	€41,596.05

The portfolio return of the account for April is *closest* to:

- 6.34%.
 - 7.68%.
 - 11.72%.
- A portfolio held 1,500 shares of BP plc common stock over the month of April 2010. Closing (i.e., final) price quotes for a BP share on the London Stock Exchange for the beginning and ending dates of the holding period were

31 March 2010	£6.234
30 April 2010	£5.755

Based only on the information given, calculate the holding period rate of return on the position in BP shares for the month of April 2010.

- 0.92%
- 7.68%
- 8.32%

Solution to 1:

C is correct.

$$R = \frac{V_1 - V_0}{V_0} = \frac{£10,200 - £10,000}{£10,000} = \frac{£200}{£10,000} = 0.02 \text{ or } 2.00\%$$

Solution to 2:

A is correct. The portfolio makes up the entire account, for which there were no external cash flows. The portfolio return can be calculated from beginning and ending market values.

$$\text{Beginning value } V_0 = €1,029,234.12 + €52,078.23 = €1,081,312.35$$

$$\text{Ending value } V_1 = €1,108,314.67 + €41,596.05 = €1,149,910.72$$

$$\begin{aligned} \text{Holding period return } R &= \frac{V_1 - V_0}{V_0} = \frac{€1,149,910.72 - €1,081,312.35}{€1,081,312.35} \\ &= 0.0634 = 6.34\% \end{aligned}$$

Solution to 3:

B is correct.

$$\text{Beginning value } V_0 = 1,500 \times £6.234 = £9,351.00$$

$$\text{Ending value } V_1 = 1,500 \times £5.755 = £8,632.50$$

$$\begin{aligned} \text{Holding period return } R &= \frac{V_1 - V_0}{V_0} = \frac{£8,632.50 - £9,351.00}{£9,351.00} \\ &= -0.0768 = -7.68\% \end{aligned}$$

Because the portfolio consists of only one security, an alternate approach would be to calculate the rate of return as a per-share value of the single security:

$$\begin{aligned} \text{Holding period return } R &= \frac{V_1}{V_0} - 1 = \frac{£5.755}{£6.234} - 1 \\ &= -0.0768 = -7.68\% \end{aligned}$$

In all the examples that follow, an equals sign (“=”) will be used at the final return calculation even though the returns have usually been rounded to the nearest basis point. It should be understood that intermediate results are calculated to full precision.

The calculation of holding period returns for a portfolio presents several practical problems:

- The choice of holding period is constrained by the availability of portfolio valuations for the beginning and the end of the holding period. It is common, for example, to calculate a monthly return based on valuations as of the last trading days of consecutive months, which might not coincide with the last calendar days of the months (as can happen when the last calendar day of a month falls on a Saturday or Sunday). The holding period is defined by the valuation dates in such cases.
- Some circumstances may lead to various assets in a portfolio reflecting differently dated values. For example, if different assets in a portfolio trade on different exchanges, the trading days might not coincide for all exchanges (because of holidays or local customs) and trading hours might differ for the various exchanges. Time zone differences in markets can also be an obstacle. Even within a given marketplace, some securities trade more frequently than others.

For the sake of accuracy, all the assets in a portfolio should be valued as close to the same time as is practicable. Differences in the timing of the prices used for valuation can be a source of uncertainty in rate-of-return calculation.

In the examples presented so far, nothing has been said about the calculation of holding period return when income from securities is received during the holding period. **Income** in this context (sometimes called **investment income**) refers to cash payments made to owners of an asset and is a source of return called the **income return**. The income return is distinct from the **price return** (or **capital gains return**), which is the return from changes in asset price. The holding period return combines both the income return and the price return. The term **total return** is often used as a synonym of holding period return to emphasize that both components of return are included in the return being reported.

For common shares, representing ownership in a corporation, income is in the form of discretionary payments known as **dividends**. For bonds, income payments are contractual obligations, the timing and amount of which are specified in the bond's contract (known as the bond's indenture); the income on bonds is known as **interest** (or **coupon interest** or coupon). (If these facts are new to you, reviewing the Appendix should be useful.)

How do we calculate a holding period rate of return when we need to account for investment income?

The simplest case arises when any income payment D occurs at the end of the holding period so that there is no question of reinvesting income. Equation 2 is appropriate in that case.

$$R = \frac{V_1 - V_0 + D}{V_0} = \frac{V_1 - V_0}{V_0} + \frac{D}{V_0} \quad (2)$$

The expression says that holding period return (or total return) is the sum of price return $[(V_1 - V_0)/V_0]$ and income return (D/V_0) . No assumptions have to be made, and there is no alternative correct expression.

To illustrate this, consider a single asset held for a single-month holding period. The beginning-of-month price was \$100, and the end-of-month price was \$106. In addition, income of \$2 was received at the end of the month. The price return is calculated as

$$\text{Price return} = \frac{V_1 - V_0}{V_0} = \frac{\$106 - \$100}{\$100} = 0.06 = 6.00\%$$

The income return is

$$\text{Income return} = \frac{D}{V_0} = \frac{\$2}{\$100} = 0.02 = 2.00\%$$

The total return is then the sum of the price and income returns:

$$\text{Total return} = \text{Price return} + \text{Income return} = 6.00\% + 2.00\% = 8.00\%$$

The total return can also be calculated independently from the price and income returns, using Equation 2:

$$\text{Total return} = \frac{V_1 - V_0 + D}{V_0} = \frac{\$106 - \$100 + \$2}{\$100} = \frac{\$8}{\$100} = 0.08 = 8.00\%$$

EXAMPLE 2**Holding Period Return When Income Is Received at End of Holding Period**

- 1 An asset was held for the month of April 2011. The end of March 2011 price was \$228.78, and the end of April 2011 price was \$226.35. In addition, income of \$10 was received at the end of April. What was the asset's total return for the month of April?
- A -1.06%
 B 3.31%
 C 4.37%
- 2 An institution lent \$1,000.00 to an individual on 1 February 2011. On 30 September 2011, the institution received \$1,080.00, including \$80.00 interest and \$1,000 capital repayment. What price return did the institution realize on the loan for the holding period 1 February–30 September 2011?
- A 0%
 B 8%
 C 12%

Solution to 1:

B is correct.

$$\text{Total return} = \frac{V_1 - V_0 + D}{V_0} = \frac{\$226.35 - \$228.78 + \$10}{\$228.78} = \frac{\$7.57}{\$228.78} = 0.0331 = 3.31\%$$

The price and income returns can also be calculated separately as follows:

$$\text{Price return} = \frac{V_1 - V_0}{V_0} = \frac{\$226.35 - \$228.78}{\$228.78} = \frac{-\$2.43}{\$228.78} = -0.0106 = -1.06\%$$

$$\text{Income return} = \frac{D}{V_0} = \frac{\$10}{\$228.78} = 0.0437 = 4.37\%$$

$$\text{Total return} = \text{Price return} + \text{Income return} = -1.06\% + 4.37\% = 3.31\%$$

Solution to 2:

A is correct.

$$\text{Price return} = \frac{\$1,000 - \$1,000}{\$1,000} = 0.0 = 0.0\%$$

The income and total returns are thus

$$\text{Income return} = \frac{\$80}{\$1,000} = 0.08 = 8.0\%$$

$$\text{Total return} = \frac{\$1,000 - \$1,000 + \$80}{\$1,000} = 0.08 = 8.0\%$$

$$\text{or Total return} = \text{Price return} + \text{Income return} = 0.0\% + 8.0\% = 8.0\%$$

If income is received during a holding period and reinvested into a portfolio (**reinvested income**), then the portfolio value at the end of the period reflects the return from the price appreciation of the beginning-of-period investment and the value of

the additional assets purchased with the reinvested income. To calculate the income return component of holding period returns, assumptions must be made regarding gains from the income component subsequent to when the income was reinvested. Usually, one of two assumptions is made. In the first case, any gains from reinvested income are included in the income return. In the second, such gains are included in the price return.

Case 1. One common assumption is that income is immediately reinvested proportionately in all the assets of a portfolio according to the percentage of the total market value of the portfolio that each asset represents. That means that nothing is assumed to change in how money is allocated among the portfolio's holdings. This assumption is typically used in constructing stock market indexes, which are notional portfolios of stocks (i.e., managed virtually on paper by the index creator). An index portfolio value is typically calculated in two ways: first, with the income reinvested in all assets of the portfolio (a **total return index**), and second, on a capital only basis (a **price index**). The portfolio values with income reinvested are used to calculate the total return of the portfolio with Equation 1:

$$R = \frac{V_1 - V_0}{V_0}$$

where V_1 includes reinvested income. The price return is

$$R = \frac{V_1^x - V_0^x}{V_0^x}$$

where the superscript x (for “ex” or “without”) indicates values without reinvested income (based on the price index). The total return minus the price return can be interpreted as the income return:

$$\frac{V_1 - V_0}{V_0} - \frac{V_1^x - V_0^x}{V_0^x} \tag{3}$$

The *income return* in this case includes the gains from reinvesting the income into the entire portfolio.

For example, consider a total return index portfolio that increases in value from 1,327.44 to 1,360.14 and an associated price return index that increases in value from 980.21 to 1,002.07 over the same period. Because the ending value of the total return index is inclusive of reinvested income, the total return for the portfolio can be calculated using Equation 1:

$$R = \frac{V_1 - V_0}{V_0} = \frac{1,360.14 - 1,327.44}{1,327.44} = 0.0246 = 2.46\%$$

The price return can be derived using values without reinvested income—in this case, a beginning price of 980.21 and an ending price of 1,002.07:

$$R = \frac{V_1^x - V_0^x}{V_0^x} = \frac{1,002.07 - 980.21}{980.21} = 0.0223 = 2.23\%$$

The total return minus the price return can be interpreted as the income return:

$$\frac{V_1 - V_0}{V_0} - \frac{V_1^x - V_0^x}{V_0^x}$$

The *income return* in this case includes the gains from reinvesting the income into the entire portfolio:

$$R = \frac{V_1 - V_0}{V_0} - \frac{V_1^x - V_0^x}{V_0^x} = 2.46\% - 2.23\% = 0.23\%$$

Case 2. The second assumption that may be made, often simpler to execute, is to include gains from reinvested income in the price return. In most portfolios, it is complicated to account for investment gains that are attributable to reinvested income apart from those of other assets in the portfolio. It is usually straightforward, however, to calculate the amount of income reinvested over a holding period. In this case, the total return R of a portfolio still can be written as Equation 1, and the decomposition into price and income return is

$$R = \frac{(V_1 - D) - V_0}{V_0} + \frac{D}{V_0} \quad (4)$$

where V_0 is the beginning market value, V_1 is the ending market value *including* the reinvested income, and D is the income received. Price return is found by deducting the income from the ending value:

$$\text{Price return} = [(V_1 - D) - V_0]/V_0$$

$$\text{Income return} = D/V_0$$

For example, let $V_0 = \text{€}100$, $V_1 = \text{€}106$, and $D = \text{€}2$. The price return, defined to include gains from reinvested income, is $[(\text{€}106 - \text{€}2) - \text{€}100]/\text{€}100 = (\text{€}104 - \text{€}100)/\text{€}100 = 0.04$ or 4 percent; the income return is $\text{€}2/\text{€}100 = 0.02$ or 2 percent; and the total return is $4\% + 2\% = 6\%$.

EXAMPLE 3

Holding Period Return Calculation When Income Is Received and Reinvested during the Holding Period

- 1 A portfolio held one asset for the month of January 2011. The beginning-of-month portfolio value was \$100 million. During the month, income of \$2 million was received and reinvested in portfolio securities. Investment gains of \$0.05 million resulted from the reinvested income. At the end of the month, the total value of the portfolio was \$108.05 million. Based only on the information given, calculate the price, income, and total returns for the month, where income return is defined to include gains from reinvested income.
- 2 A portfolio held one asset for the month of April 2011. The beginning-of-month portfolio value was \$228.78 million. During the month, income of \$10 million was received, and interest on that \$10 million was earned for the rest of the month. At the end of the month, the total value of the portfolio was \$236.85 million. Based only on the information given, calculate the price, income, and total returns for the month, where price return is defined to include gains from reinvested income.

Solution to 1:

The income received is already included in the portfolio's ending value and, therefore, does not need to be added back when calculating the total return.

$$\text{Total return} = \frac{V_1 - V_0}{V_0} = \frac{\$108.05 - \$100}{\$100} = \frac{\$8.05}{\$100} = 0.0805 = 8.05\%$$

The income return, defined to include gains from reinvested income, can be calculated directly as

$$\frac{\$2 + \$0.05}{\$100} = \frac{\$2.05}{\$100} = 0.0205 = 2.05\%$$

Therefore, the price return was 6 percent: Price return = Total return – Income return = 8.05% – 2.05% = 6.00%.

Solution to 2:

The income received is already included in the portfolio's ending value and, therefore, does not need to be added back when calculating the total return.

$$\text{Total return} = \frac{V_1 - V_0}{V_0} = \frac{\$236.85 - \$228.78}{\$228.78} = \frac{\$8.07}{\$228.78} = 0.0353 = 3.53\%$$

The price return, defined to include gains from reinvested income, is

$$\begin{aligned} \text{Price Return} &= \frac{V_1 - V_0 - D}{V_0} \\ &= \frac{\$236.85 - \$228.78 - \$10}{\$228.78} \\ &= \frac{-\$1.93}{\$228.78} = -0.0084 = -0.84\% \end{aligned}$$

The income return is

$$\text{Income return} = \frac{D}{V_0} = \frac{\$10}{\$228.78} = 0.0437 = 4.37\%$$

We can check that the sum of the price and income returns equals the total return previously calculated: Total return = Price return + Income return = –0.84% + 4.37% = 3.53%.

In this example, the income return reflects only the income received (\$10 million) and not any subsequent gain from investing the income. Any gains from the reinvested income are *included in the price return*.

2.1.1 Common Shares

When a dividend is announced, a certain date is set—the **ex-dividend date**—from which point forward a purchaser of a share is not entitled to the dividend payment. The trading day before the ex-dividend day is the last one in which an investor can purchase a share so as to appear as the owner of record as of a date called the record date; only owners as of that date are entitled to receive the dividend.² The date at which the dividend amount is actually paid—the **pay date**—can be weeks later.

Consider a portfolio with beginning and ending values V_0 and V_1 , respectively, consisting initially of a single share of common stock. If the portfolio holds the stock over a period starting prior to the ex-dividend date and ending after the ex-dividend date but before the pay date and if the portfolio is valued after the ex-dividend date, the value of the dividend should be included in the portfolio value as accrued income when calculating total return because ownership of the dividend remains with the owner at the time the stock went ex-dividend. The expressions given for price and income return are then applicable.

² The **record date** may be defined as the date on which an investor needs to be on record as an owner of a share to receive a dividend that has been declared on it. Most commonly, payment for a share purchase (as for bond purchases) is made three business days after the transaction date (“ $T + 3$ settlement”), and the stock's record date is thus commonly two business days after the ex-dividend date.

EXAMPLE 4**Calculation of Income Return, Price Return, and Total Return for a Stock**

Exxon Mobil Corporation (XOM) announced a dividend of \$0.44 per share in February 2011. The ex-dividend date was 8 February 2011, and the pay date was 10 March 2011. Closing prices of XOM on the New York Stock Exchange were

Date	Closing Price
31 January 2011	\$80.680
7 February 2011	\$83.930
8 February 2011	\$82.980
28 February 2011	\$85.530

Calculate the income return, price return, and total return on holding XOM from the close of trading on the 7 February through 8 February 2011 (assuming purchase at the close of trading on 7 February 2011) and for the month of February.

Solution:

The closing price of XOM on 8 February is ex-dividend—that is, the value of owning a share without owning the dividend. The ending value of the portfolio should include the value of the dividend even though it had not yet been paid. Thus, we use portfolio value as given in the last column in calculating return. Because the pay date is after 28 February, the issue of reinvestment of the dividend does not arise.

	Closing Price	Dividend Recognized on Ex-Dividend Date		Portfolio Value
		Forward		
31 January 2011	\$80.680			\$80.680
7 February 2011	\$83.930			\$83.930
8 February 2011	\$82.980	\$0.44		\$83.420
28 February 2011	\$85.530	\$0.44		\$85.970

For 8 February,

$$\begin{aligned} \text{Price return} &= \frac{(V_1 - D) - V_0}{V_0} = \frac{(\$83.42 - \$0.44) - \$83.93}{\$83.93} \\ &= -0.0113 = -1.13\% \end{aligned}$$

$$\text{Income return} = \frac{D}{V_0} = \frac{\$0.44}{\$83.93} = 0.0052 = 0.52\%$$

$$\begin{aligned} \text{Total return} &= \frac{(V_1 - D) - V_0}{V_0} + \frac{D}{V_0} = \frac{(\$83.42 - \$0.44) - \$83.93}{\$83.93} + \frac{\$0.44}{\$83.93} \\ &= -0.0061 = -0.61\% \end{aligned}$$

Written differently, Total return = Price return + Income return = $-1.13\% + 0.52\% = -0.61\%$.

For the month of February,

$$\text{Price return} = \frac{(V_1 - D) - V_0}{V_0} = \frac{(\$85.97 - \$0.44) - \$80.68}{\$80.68} = 0.0601 = 6.01\%$$

$$\text{Income return} = \frac{D}{V_0} = \frac{\$0.44}{\$80.68} = 0.0055 = 0.55\%$$

$$\begin{aligned} \text{Total return} &= \frac{(V_1 - D) - V_0}{V_0} + \frac{D}{V_0} = \frac{(\$85.97 - \$0.44) - \$80.68}{\$80.68} + \frac{\$0.44}{\$80.68} \\ &= 0.0656 = 6.56\% \end{aligned}$$

Written differently, Total return = Price return + Income return = 6.01% + 0.55% = 6.56%.

2.1.2 Portfolio Return Calculated from Returns to Individual Holdings

How do we calculate the holding period return on a portfolio consisting of many assets when the information given consists only of data for the individual assets? If we can calculate the returns on the individual assets, the return on a portfolio that buys and holds the assets can be calculated as a weighted average of the assets' returns, using the beginning-of-period portfolio weights for the assets. Let R_i be the return on asset i and w_i be the portfolio weight of asset i —that is, $w_i = (\text{beginning-of-period market value of asset } i) / (\text{beginning-of-period market value of portfolio containing asset } i)$ —so that the sum of portfolio weights equals 1. Denoting by R_p the return on the portfolio,

$$R_p = \sum_{i=1}^n w_i R_i \quad (5)$$

Thus, if the portfolio consists of three assets with respective weights of 0.20, 0.50 and 0.30 and returns of 2%, 6%, and 12%, then the portfolio return is 7%, calculated as follows:

$$R_p = \sum_{i=1}^3 w_i R_i = w_1 R_1 + w_2 R_2 + w_3 R_3 = (0.20)(2\%) + (0.50)(6\%) + (0.30)(12\%) = 7\%$$

EXAMPLE 5

Calculating Portfolio Return from the Returns of Component Assets

A portfolio consists of four assets with respective portfolio weights of 0.40, 0.25, 0.20, and 0.15. Over the year, the assets achieved returns of -10%, 4%, 20%, and 24%, respectively. The portfolio's rate of return was closest to:

- A 4.6%.
- B 9.5%.
- C 9.8%.

Solution:

A is correct. $0.40(-10\%) + 0.25(4\%) + 0.20(20\%) + 0.15(24\%) = 4.6$ percent.

2.1.3 Bonds

The return calculations given thus far have been illustrated using common shares. There are important distinctions between the income on common shares (dividends) and the income on bonds (interest). A common stock investor receives a declared dividend if he or she appears as the owner of record on a stated date (the so-called record date for receipt of the dividend); otherwise, whatever the length of time the investor held the shares, no part of the declared dividend is received. In contrast, when

a bond is sold, the seller receives the price quoted in the transaction plus a share of the upcoming interest payment, known as **accrued interest** (AI), which is proportional to the fraction of days between interest payments that the bond has been held by the seller.³ For example, if a \$1,000 face value bond with \$20 semi-annual interest payments is sold for \$960 exactly halfway between interest payment dates, the seller would receive $\$960 + (1/2)\$20 = \$970$. The \$10 difference is accrued interest: interest that has been earned by the bond seller (but not yet paid by the bond issuer) since the last interest payment. Conventional price quotes exclude the value of coupon interest accrued to the date of the quote. The price without accrued interest is called the **clean price**; here, the clean price is \$960. The price with accrued interest, here \$970, is sometimes called the **dirty price**. The dirty price represents the value at which the asset can be exchanged for other assets in the market.

The price with accrued interest is the relevant one for calculating the rate of return on a bond using Equation 1. Adapted to bonds and, for simplicity, limiting ourselves to the case in which the bond is purchased and sold within a single period between interest payments, we obtain⁴

$$\begin{aligned} \text{Total return} &= \frac{(\text{Ending dirty price} - \text{Beginning dirty price})}{\text{Beginning dirty price}} \\ &= \frac{\left[\begin{array}{l} (\text{Ending clean price} - \text{Beginning clean price}) \\ + \text{Change in accrued interest} \end{array} \right]}{\text{Beginning dirty price}} \end{aligned} \quad (6a)$$

where

$$\text{Change in AI} = \text{Ending AI} - \text{Beginning AI}^5$$

$$\text{Price return} = \frac{(\text{Ending clean price} - \text{Beginning clean price})}{\text{Beginning dirty price}} \quad (6b)$$

$$\text{Income return} = \frac{\text{Change in accrued interest}}{\text{Beginning dirty price}} \quad (6c)$$

Assume the bond just described is purchased halfway between interest payments for \$970 (including accrued interest). The bond is sold three-fourths of the way into the six-month period for \$950 plus accrued interest of $(3/4)(\$20) = \15 . The amount realized on the sale is $\$950 + \$15 = \$965$.

The holding period return for the period beginning with the date the bond was purchased at a cost of \$970 and ending on the date it was sold for total proceeds of \$965 is

$$(\$965 - \$970)/\$970 = -0.52\%$$

The price return is $(\$950 - \$960)/\$970 = -1.03$ percent. Based on a change of accrued interest of $\$15 - \$10 = \$5$, the income return is $\$5/\$970 = 0.52$ percent. The two return components sum to -0.52 percent (apart from rounding error).

³ When a bond is purchased, one pays AI counting from the last coupon payment date through the settlement date of the trade. The details of accrued interest calculations are beyond the scope of this reading and, in fact, differ across markets. One complication is that different markets make different assumptions about the number of days in a month and/or a year when calculating accrued interest (e.g., the US bond market assumes 30 days in all months and 360 days in a year). Some bonds, such as those in default, trade without accrued interest (they are said to trade “flat”).

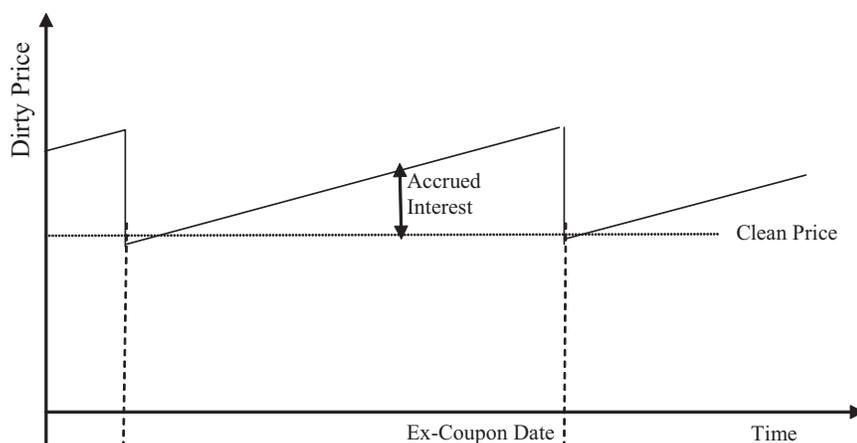
⁴ For longer periods, one would need to account for the receipt and reinvestment of coupon payments. Note that the income return shown in Equation 6c does not have a coupon payment component.

⁵ That is, the accrued interest received if the bond is sold at the end of the holding period minus the accrued interest paid when the bond was purchased at the beginning of the holding period.

Bonds, like common shares, can be thought of as having an “ex-date,” or **ex-coupon date**: The entire coupon will be paid to the investor owning the bond on the day before the ex-coupon date. However, whereas for common shares the record date and the ex-dividend date are typically somewhat arbitrary,⁶ for specific types of bonds the ex-coupon date is often set predictably in relation to trade settlement periods. For example, US corporate bond trades typically settle in three business days (e.g., on Thursday if the trade is on Monday, assuming Monday through Thursday are all days that are open for trading). If that Thursday is the coupon payment date, the Monday is effectively the ex-coupon date because a buyer on Monday does not receive the upcoming interest payment and the transaction is without accrued interest.⁷

Exhibit 1 shows the relationship between clean and dirty prices and accrued interest over time assuming the clean price remains constant over the period.

Exhibit 1 Relationship between Price and Accrued Interest for a Bond



EXAMPLE 6

Calculation of Holding Period Return for a Bond

A \$1,000 face value, 4 percent coupon, semi-annual pay bond is issued on 1 January and purchased at par. The first coupon will be paid on 1 July. The bond is quoted at a clean price of \$1,020 on 31 January.

- 1 Calculate the rate of return on the bond for January.
- 2 Calculate the price return on the bond for January.
- 3 Calculate the income return on the bond for January.

Solution to 1:

Because the bond was held for one month, 1/6 of the semi-annual coupon payment needs to be accrued.

$$\text{Semi-annual coupon} = \$1,000 \times 4\% \times 0.5 = \$20$$

⁶ For equities, the period between the ex-dividend date and the payment date is typically longer.

⁷ Trades carried out in a longer ex-coupon period (the ex-coupon date through the payment date) are often settled on the basis of *subtracting* interest from the clean price. Details vary by market; for example, gilts (UK government bonds) usually go ex-coupon seven days before the payment date.

$$\text{Accrued interest} = \$20 \times (1/6) = \$3.33$$

Using Equation 6a,

$$\text{Total return} = \frac{\$1,020 - \$1,000 + \$3.33}{\$1,000} = 0.0233 = 2.33\%$$

Solution to 2:

Using Equation 6b,

$$\text{Price return} = \frac{\$1,020 - \$1,000}{\$1,000} = 0.0200 = 2.00\%$$

Note that the dirty and clean prices are the same on 1 January because there is no accrued interest.

Solution to 3:

The income return accounts for the portion of the coupon payment that has been earned but not yet received. Using Equation 6c,

$$\text{Income return} = \frac{\$3.33}{\$1,000} = 0.0033 = 0.33\%$$

EXAMPLE 7

Calculation of Income Return, Price Return, and Total Return for a Bond

A \$1,000 face value, 4 percent coupon, semi-annual pay bond is issued on 1 January and purchased at par. The first coupon will be paid on 1 July. The bond is quoted on each of the following dates at the clean prices noted:

31 March at \$1,040

31 May at \$1,055

Calculate the holding period return for the bond including the price and income return components for the period 31 March to 31 May.

Solution:

$$\text{Accrued interest to 31 March} = \$20 \times (3/6) = \$10.00$$

$$\text{Accrued interest to 31 May} = \$20 \times (5/6) = \$16.67$$

Change in accrued interest: \$6.67

$$\text{Dirty price on 31 March} = \$1,040 + \$10 = \$1,050 = V_0$$

$$\text{Total return} = \frac{\$1,055 - \$1,040 + \$6.67}{\$1,050} = 0.0206 = 2.06\%$$

$$\text{Price return} = \frac{\$1,055 - \$1,040}{\$1,050} = 0.0143 = 1.43\%$$

$$\text{Income return} = \frac{\$6.67}{\$1,050} = 0.0064 = 0.64\%$$

2.1.4 Cash in Portfolio Performance

In a typical portfolio, such as a brokerage account, there is usually a provision for holding money that is not otherwise invested in securities in cash and cash equivalents. Such investments may be called the **liquidity reserve** asset, or simply cash. Income from dividends or coupon interest and sales of securities typically increase the value of the liquidity reserve, whereas purchases of securities typically decrease the liquidity reserve. When money is deposited to or withdrawn from the account, the money flows into or out of the liquidity reserve.

EXAMPLE 8

Value of a Liquidity Reserve

On 31 March, an investor has a portfolio consisting of 9,000 shares of stock and \$10,000 cash. The stock has a dividend receivable of \$0.10 per share; its pay date is 30 April. The stock's 31 March price is \$10 per share, and the cash earns interest of 1.2 percent annually (10 bps per month). At the end of the month, the share price of the stock is \$10.50. Consider the receivable dividend as part of the liquidity reserve. Calculate the ending values of the stock holdings, the liquidity reserve, and the total portfolio, along with the returns of the portfolio and its components for the month.

Solution:

The cash asset earns 10 bps on the \$10,000 value, and the \$900 receivable dividend earns no interest until it is paid. The beginning and ending values of the portfolio components are

	31 March	30 April	Return
Stock	9,000 × \$10 = \$90,000	9,000 × \$10.50 = \$94,500	5.00%
Liquidity reserve:			
Cash asset	\$10,000	(\$10,000 × 1.001) + \$900 = \$10,910	
Receivable dividend	9,000 × \$0.10 = \$900	\$0	
Liquidity reserve total	\$10,900	\$10,910	0.09%
Portfolio total	\$100,900	\$105,410	4.47%

Note that although the cash asset pays 1.2 percent annually, the return of the liquidity reserve was 9 bps for the month instead of 10 bps because cash from the dividend was not received until the end of the month.

The value of the liquidity reserve is an important component in calculating portfolio performance, and care should be taken in its accounting. (Note that the Global Investment Performance Standards [GIPS®] require returns from cash and cash equivalents held in portfolios to be included in all return calculations [CFA Institute 2010].)

2.1.5 Returns on Foreign Assets

A **domestic asset** is an asset that trades in the investor's home currency (the one he or she typically uses for spending). An example is a US government bond from the perspective of a US-domiciled investor. This same bond is a non-domestic asset from a eurozone investor's perspective: *Domestic* refers to a relation between the asset and

the investor, not an inherent property of the asset. **Foreign assets** (or **non-domestic assets**) are assets denominated in currencies other than the investor's domestic (base) currency. The **domestic currency** (also called **base currency** or **home currency**) is the currency in which the portfolio is valued for return calculations because consumption takes place chiefly in that currency. Suppose an investor holds a foreign asset. In the (foreign) market in which the asset trades, the asset goes up in value by 10 percent (as measured in that market). From this investor's perspective, 10 percent is the asset's return in foreign currency terms—its **foreign-currency return**. This return is often called the **local currency return** (or local return) because it is the return of the asset expressed in local-currency terms, where "local" refers to that asset's market. The investor can restate the foreign-currency return in home-currency terms—that is, as a **domestic-currency return**. The domestic-currency return would reflect the foreign-currency return and any currency exchange rate gains or losses over the holding period.

To value a foreign-currency-denominated asset in an investor's base currency, the **spot exchange rate** is required, which is the rate at which foreign currency can be exchanged for immediate delivery of the base currency. If V^{FC} is the value of an asset in a foreign currency (FC) and $S^{d/f}$ is the spot exchange rate measured as the number of domestic-currency units required to buy a unit of foreign currency—indicated by the superscript d/f —then the value V of the asset in the domestic currency is $V^{FC}S^{d/f}$.⁸ For example, if the value of an asset in a foreign currency (V^{FC}) is 2.5678 and the spot exchange rate ($S^{d/f}$) is 0.6982, then the value in the domestic currency is $2.5678 \times 0.6982 = 1.7928$.

Using subscripts 0 and 1 to denote the beginning and ending times of a holding period, respectively, the base-currency return of a foreign-currency asset can be written as

$$R = \frac{V_1}{V_0} - 1 = \frac{V_1^{FC} S_1^{d/f}}{V_0^{FC} S_0^{d/f}} - 1 \quad (7)$$

Building off the previous example, if the value of the asset in the foreign currency (V_1^{FC}) changed to 2.7837 and the spot exchange rate ($S_1^{d/f}$) moved to 0.6876, applying Equation 7, the holding period return would be

$$R = \frac{V_1^{FC} S_1^{d/f}}{V_0^{FC} S_0^{d/f}} - 1 = \frac{(2.7837)(0.6876)}{(2.5678)(0.6982)} - 1 = \frac{1.9141}{1.7928} - 1 = 0.0676 = 6.76\%$$

In the local currency, the return can be calculated as $R^{FC} = (V_1^{FC} / V_0^{FC}) - 1$; as mentioned earlier, this return can be referred to either as the foreign-currency or local-currency return or just the local return. The **currency return** is defined as $R^C = (S_1^{d/f} / S_0^{d/f}) - 1$; it represents the percentage change in the exchange rate over the period being considered (equivalently, the return from holding one unit of foreign currency). The return of an asset in base currency terms shown in Equation 7 can now be expressed as

$$R = (1 + R^{FC})(1 + R^C) - 1 \quad (8a)$$

or

$$R = R^{FC} + R^C + (R^{FC})(R^C) \quad (8b)$$

⁸ An exchange rate quotation of the form "domestic-currency units per foreign-currency unit" is known as a **direct quotation**; exchange rates can also be quoted in terms of "foreign-currency units per domestic-currency unit" which is an **indirect quotation**.

Equation 8a shows the base-currency return on an investment denominated in a foreign currency as the product of one plus the investment's local return and one plus the currency return. Equation 8b carries through the multiplication and shows the base-currency return as the sum of the local return, the currency return, and the cross-product between the local and currency returns.

To illustrate, suppose a eurozone investor makes a \$100 million investment that, in one year's time, is worth \$110 million. The US dollar is the foreign currency from this investor's perspective: $V_0^{FC} = \$100$ and $V_1^{FC} = \$110$. In foreign-currency (US dollar) terms, the return is 10 percent:

$$R^{FC} = \left(V_1^{FC} / V_0^{FC} \right) - 1 = \$110 / \$100 - 1 = 0.10 = 10.00\%.$$

When the investment is made, €0.7 buys \$1 (one can write the exchange rate as €0.7/\$1). One year later, the dollar has appreciated against the euro; in other words, a given amount of dollars buys more euros or, equivalently, the euro cost of \$1 is greater. Suppose, in particular, that the exchange rate has increased to €0.735/\$1. Thus, $S_0^{d/f} = 0.70$ and $S_1^{d/f} = 0.735$ and the currency return can be computed as 5%:

$$R^C = \left(S_1^{d/f} / S_0^{d/f} \right) - 1 = (0.735/0.70) - 1 = 0.05 = 5\%$$

According to Equation 7,

$$R = \frac{V_1^{FC} S_1^{d/f}}{V_0^{FC} S_0^{d/f}} - 1 = \frac{\$110 \times \text{€}0.735/\$1}{\$100 \times \text{€}0.7/\$1} - 1 = \frac{\text{€}80.85}{\text{€}70} - 1 = 0.1550 = 15.5\%$$

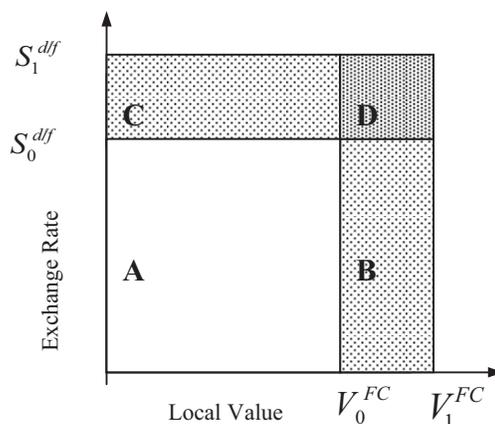
Equivalently, using Equation 8a:

$$R = (1 + R^{FC})(1 + R^C) - 1 = (1.10)(1.05) - 1 = 0.1550 = 15.5\%$$

The return of 15.5 percent in the investor's home currency exceeds the investment's local return of 10 percent. What has happened? The investor has benefited by the appreciation of the US dollar against the euro because in converting dollars to euros at $t = 1$, a dollar buys more euros than before.

The impact of currency return can also be analyzed in monetary terms, as shown in Exhibit 2.

Exhibit 2 Impact of Currency on Returns



Area A is the initial value: $V_0^F S_0^{d/f}$.

Area B is the gain from the local return only: $(V_1^{FC} - V_0^{FC})S_0^{d/f}$.

Area C is the gain from currency return only: $V_0^{FC}(S_1^{d/f} - S_0^{d/f})$.

Area D is the gain from the interaction of both the local return and currency returns and is a consequence of both. Area D is found by subtracting areas A, C, and B from the total area $V_1^{FC}S_1^{d/f}$:

$$\begin{aligned} & V_1^{FC}S_1^{d/f} - V_0^{FC}S_0^{d/f} - V_0^{FC}(S_1^{d/f} - S_0^{d/f}) - (V_1^{FC} - V_0^{FC})S_0^{d/f} \\ &= (V_1^{FC} - V_0^{FC})(S_1^{d/f} - S_0^{d/f}) \end{aligned}$$

Using information from the previous example,

$$\text{Area A} = (\$100 \times \text{€}0.7/\text{\$}1) = \text{€}70$$

$$\text{Area B} = (\$110 - \$100)(\text{€}0.7/\text{\$}1) = \text{€}7$$

$$\text{Area C} = \$100(\text{€}0.735/\text{\$}1 - \text{€}0.7/\text{\$}1) = \text{€}3.5$$

$$\text{Area D} = \text{€}80.85 - \text{€}70 - \text{€}3.5 - \text{€}7 = \text{€}0.35$$

where, in the calculation of Area D, $\text{€}80.85 = \$110(\text{€}0.735/\text{\$}1)$ is the ending value of the asset in euros.

Note that $B + C + D$, the total gain ($\text{€}10.85$), equals the first two terms in the expression for D, $V_1^{FC}S_1^{d/f} - V_0^{FC}S_0^{d/f}$. Therefore, the base-currency rate of return equals

$$\frac{V_1^{FC}S_1^{d/f} - V_0^{FC}S_0^{d/f}}{V_0^{FC}S_0^{d/f}} = \frac{V_1^{FC}S_1^{d/f}}{V_0^{FC}S_0^{d/f}} - 1 = \frac{\text{€}80.85}{\text{€}70} - 1 = 15.5\%$$

which is Equation 7 above.

The base-currency rate of return can also be derived by dividing the total gain ($B + C + D$) by the initial value (A).

$$R = \frac{\text{€}7 + \text{€}3.5 + \text{€}0.35}{\text{€}70} = \frac{\text{€}10.85}{\text{€}70} = 0.155 = 15.5\%$$

This expression shows that the interaction effect $\text{€}0.35/\text{€}70 = 0.5\%$ explains why the base-currency return is slightly higher than the sum of the local and currency returns ($10\% + 5\%$).

If a portfolio consists of a collection of assets in multiple currencies, all the assets must be converted to the base currency using the respective spot exchange rates in order to calculate a base currency return. Once a portfolio return has been calculated in a currency, by using Equation 7 one can convert the return to any other currency.

EXAMPLE 9

Holding Period Return for a Portfolio of Assets in Multiple Currencies

A portfolio holds assets in US dollars (\$), UK sterling (£), and Japanese yen (¥). Calculate performance in the base currency of euros (€), and then convert this return into \$, £, and ¥ returns.

	31 March	30 April
US equities	\$56.7 million	\$63.4 million
UK equities	£23.5 million	£26.7 million
Japanese equities	¥1,500 million	¥1,350 million

(continued)

	31 March	30 April
One US dollar buys this many euros: ^a	0.75	0.70
One UK pound sterling buys this many euros:	1.20	1.30
One Japanese yen buys this many euros:	0.009091	0.010000

^a Another way to express the same thing is that this is the euro cost of one US dollar.

Solution:

Calculate total portfolio beginning and ending values in euros by multiplying each group of assets by the applicable exchange rate.

$$\text{Beginning value: } (\$56.7 \times 0.75) + (£23.5 \times 1.20) + (¥1,500 \times 0.009091) = €84.36$$

$$\text{Ending value: } (\$63.4 \times 0.7) + (£26.7 \times 1.30) + (¥1,350 \times 0.010000) = €92.59$$

$$\text{Portfolio return in } € = \frac{92.59}{84.36} - 1 = 9.76\%$$

The base-currency return in € can be converted into any other currency return by applying the appropriate currency return:

$$\text{Portfolio return in } \$ = (1.0976) \left(\frac{0.75}{0.70} \right) - 1 = (1.0976)(1.0714) - 1 = 17.6\%$$

$$\text{Portfolio return in } £ = (1.0976) \left(\frac{1.20}{1.30} \right) - 1 = (1.0976)(0.9231) - 1 = 1.32\%$$

$$\begin{aligned} \text{Portfolio return in } ¥ &= (1.0976) \left(\frac{0.009091}{0.010000} \right) - 1 = (1.0976)(0.909100) - 1 \\ &= -0.22\% \end{aligned}$$

Alternatively, the return in each currency can be obtained by recalculating the beginning and ending values in each currency as follows:

$$\text{Beginning value in } \$: \$56.7 + £23.5 \times \frac{1.2}{0.75} + ¥1,500 \times \frac{0.009091}{0.75} = \$112.48$$

$$\text{Ending value in } \$: \$63.4 + £26.7 \times \frac{1.3}{0.7} + ¥1,350 \times \frac{0.01}{0.7} = \$132.27$$

$$\text{Portfolio return in } \$: \frac{\$132.27}{\$112.48} - 1 = 17.6\%$$

$$\text{Beginning value in } £: \$56.7 \times \frac{0.75}{1.2} + £23.5 + ¥1,500 \times \frac{0.009091}{1.2} = £70.30$$

$$\text{Ending value in } £: \$63.4 \times \frac{0.7}{1.3} + £26.7 + ¥1,350 \times \frac{0.01}{1.3} = £71.22$$

$$\text{Portfolio return in } £: \frac{£71.22}{£70.30} - 1 = 1.31\%$$

Beginning value in

$$¥: \$56.7 \times \frac{0.75}{0.009091} + £23.5 \times \frac{1.2}{0.009091} + ¥1,500 = ¥9,279.67$$

$$\text{Ending Value in ¥: } \$63.4 \times \frac{0.7}{0.01} + £26.7 \times \frac{1.3}{0.01} + ¥1,350 = ¥9,259$$

$$\text{Portfolio return in ¥: } \frac{¥9,259}{¥9,279.67} - 1 = -0.22\%$$

The dollar return looks higher because the dollar fell against the euro, but both the pound and yen appreciated against the euro, generating lower returns in these base currencies.

2.1.6 Short and Leveraged Positions

Although we have defined a portfolio as a collection of assets, the concept of return can be extended to a collection of liabilities, which may be thought of as “negative assets.” Liabilities can be modeled similarly to assets sold short, or “short holdings.” The return of a portfolio consisting only of liabilities is sometimes called a **liability return**. The return of a portfolio of both assets and liabilities is sometimes called a **surplus return**. When a portfolio consists of both long and short positions, care must be taken to segment the assets into those held long and those held short. It is important to maintain the convention that if an asset falls in value, then the return on that individual asset is negative whether the holding is long or short. It is the combination of a negative holding with a negative return that results in a profit.

Consider the following data for the assets and liabilities of a pension fund:

Pension Assets

Beginning value on 1 January 2010	€100
Change in value in 2010	7%

Pension Liabilities

Beginning value on 1 January 2010	€60
Change in value in 2010	15%

Note: Euro values are in millions.

The fund’s beginning surplus would be calculated by subtracting the value of liabilities from the value of assets: €100 – €60 = €40. The ending surplus is €100(1.07) – €60(1.15) = €107 – €69 = €38. The surplus return is calculated as the percentage change in surplus: (€38 – €40)/€40 = –0.05 or –5 percent.

The following example illustrates the calculation of return for a portfolio with long and short positions.

EXAMPLE 10**Calculating Returns for Portfolios with Long and Short Positions**

A portfolio holds two long shares of Stock A, valued at \$25 per share. In addition, the portfolio sells short one share of Stock B, valued at \$30 per share. One month later, both stocks A and B are valued at \$28 per share. Based only on the facts given, calculate the one-month returns on the long position, the short position, and the total portfolio.⁹

Solution:

The long-position, short-position, and total portfolio returns are 12 percent, -6.67 percent, and 40 percent, respectively. The total portfolio return can be arrived at in one of two ways: as a weighted average of long- and short-position returns or by netting long and short beginning and ending amounts.

1 Weighted average of long- and short-position returns:

Long-position return

$$\begin{aligned}\text{Beginning value} &= \$25 \times 2 = \$50 \\ \text{Ending value} &= \$28 \times 2 = \$56.\end{aligned}$$

Thus, the return on the long position is

$$R = \frac{\$56 - \$50}{\$50} = \frac{\$6}{\$50} = 0.12 = 12.00\%$$

Short-position return

The return on the short position is

$$R = \frac{\$28 - \$30}{\$30} = \frac{-\$2}{\$30} = -0.0667 = -6.67\%$$

The initial equity is the amount invested in the long position minus the amount invested in the short position: $\$50 - \$30 = \$20$. The long position represents 250 percent of the beginning portfolio equity ($\$50/\$20 = 2.5$), and the short position represents -150 percent ($-\$30/\$20 = -1.5$). Multiplying the returns by the respective position weights yields the total portfolio return:

$$R = (12.00\% \times 2.5) + (-6.67\% \times -1.5) = 30\% + 10\% = 40\%$$

2 Netting long- and short-position amounts:

The initial equity, as already noted, is $\$50 - \$30 = \$20$. This is the denominator. The long position gained $\$56 - \$50 = \$6$. From this we subtract the gain on the short position: $\$28 - \$30 = -\$2$ (i.e., a negative gain); thus the numerator reduces to $\$6 - (-\$2) = \$6 + \$2 = \$8$.

$$R = \frac{(\$56 - \$50) - (\$28 - \$30)}{(\$50 - \$30)} = \frac{\$8}{\$20} = 0.40 = 40\%$$

⁹ In practice, factors arise that complicate the rate of return calculation. For example, the short seller may have to deposit collateral over and above the short sale proceeds when selling short. Short sellers sometimes negotiate for short sale proceeds to earn interest. For the sake of simplicity, these and other complicating factors are ignored here.

Leverage is the use of borrowing¹⁰ or various financial instruments to magnify potential gains and losses relative to the investor's equity position. The **leveraged return** (or **levered return**) represents the actual return achieved based on the capital invested—that is, on the leveraged (levered) equity. The **cash basis return** represents the unleveraged return on the underlying investment—that is, the return on total capital invested including the amount financed by borrowing.

The successful use of leverage involves earning a cash basis return that is in excess of the cost of borrowed funds. This result can most easily be shown by an example.

Suppose an investor has equity capital of \$400 million to invest. The investor borrows \$100 million at an interest rate (i) of 5 percent per year; thus, in one year, interest of $5\% \times \$100$ million = \$5 million is payable. In total, after borrowing, the investor has \$500 million of total capital invested. The rate of return earned on this sum is the cash basis return, denoted R_{CB} .

Suppose \$40 million is earned on the \$500 million in one year. Then, $R_{CB} = \$40/\$500 = 0.08$ or 8%. If R_{LE} is the return on levered equity, $R_{LE} = \$35/\$400 = 0.0875$ or 8.75%. The numerator is \$40 million minus the \$5 million cost of borrowing, or \$35 million. The denominator in the expression is \$400 million, the equity in the investment.

A simple expression is available for the return on the equity in a portfolio that uses leverage. Defining the leverage ratio as $L = \text{Market value of margin debt}/(\text{Market value of margin debt} + \text{Market value of equity})$, the expression is

$$R_{LE} = \frac{R_{CB} - iL}{1 - L} \quad (9)$$

In the case presented, $L = \$100/(\$100 + \$400) = 0.20$. Thus,

$$R_{LE} = \frac{R_{CB} - iL}{1 - L} = \frac{0.08 - 0.05(0.2)}{1 - 0.20} = \frac{0.07}{0.80} = 0.0875 = 8.75\%$$

which confirms the result found. In Equation 9, note that as the use of debt relative to equity increases, the leverage ratio becomes larger and $1 - L$ becomes smaller, which tends to magnify returns.¹¹

EXAMPLE 11

Leveraged and Cash Basis Returns

A portfolio purchased 100 shares of common stock valued at \$48 per share. Half the purchase price was paid with cash, while the remainder was paid for using borrowed money. One month later, the stock appreciated to \$50 per share. The annual interest rate on borrowing is 2 percent. Calculate the cash basis and leveraged returns for the investment over the one-month holding period.

Solution:

$$\text{Initial equity exposure} = 100 \times \$48 = \$4,800$$

$$\text{Final equity exposure} = 100 \times \$50 = \$5,000$$

$$\text{Net profit} = \$5,000 - \$4,800 = \$200$$

$$\text{Cash basis return } (R_{CB}) = \frac{\$200}{\$4,800} = 0.0417 = 4.17\%$$

$$\text{Initial margin} = \$4,800/2 = \$2,400$$

¹⁰ The use of borrowing in this context is sometimes called the use of **margin**.

¹¹ The denominator effect tends to outweigh the effect of iL in the numerator of Equation 9.

$$\text{Leverage ratio } (L) = \$2,400/\$4,800 = 0.50$$

$$\begin{aligned} \text{Leveraged return } (R_{LE}) &= \frac{\$200 - [\$2,400 \times (0.02/12)]}{\$2,400} = \frac{\$196}{\$2,400} \\ &= 0.0817 = 8.17\% \end{aligned}$$

Equivalently, using Equation 9,

$$\begin{aligned} R_{LE} &= \frac{R_{BC} - iL}{1 - L} = \frac{0.0417 - [(0.5) \times (0.02/12)]}{0.5} = \frac{0.04087}{0.5} \\ &= 0.0817 = 8.17\% \end{aligned}$$

2.1.7 Accounting and Trade Considerations

We have defined rate of return as the change in value of a portfolio. **Market value** is the price at which investors can buy or sell an investment at a given time multiplied by the quantity held plus any accrued income. Market value provides the clearest indication of an investor's true economic wealth. Other accounting measures of value are useful for various purposes, but returns calculated for performance evaluation are almost always calculated from market values.

2.1.7.1 Book Value vs. Market Value In the context of a securities portfolio, the accounting concept of **book value** (sometimes also called **cost value** or **cost basis**) is the value of an asset recorded at the time it was purchased. Because market prices change over time, market values of assets diverge from book values over time. Moreover, accounting conventions (such as average cost basis accounting) can cause book values to change over time. For example, if 10 shares of stock were purchased at \$10 per share and later 10 more shares were purchased at \$14 per share, the cost value under average cost accounting would be \$240, or \$12 per share. Although a return can be calculated from book values— $(12 - 10)/10$ or 20% from the time of the first purchase to the time of the second purchase in this example—such a return may not accurately reflect the change in the owner's wealth. The return based on market values in the example is $(14 - 10)/10$, or 40%.

2.1.7.2 Fair Value vs. Market Value Fair value is the amount at which an investment could be exchanged in a current arm's-length transaction between willing parties in which the parties each act knowledgeably and prudently. In the case of liquid securities, fair value and market value will typically be the same. Where they will diverge is in the case of illiquid or infrequently traded investments where a market price is not readily available or is based on a transaction that occurred so long ago that its relevance is questionable (a stale price). In such a case, assumptions often must be made to arrive at the best estimate of fair value. It should be noted that beginning 1 January 2010, the GIPS standards require portfolio valuations to be based on fair value.

2.1.7.3 Realized and Unrealized Gains and Losses If an asset's market value has changed from its book value (which, in the simplest case, might be thought of as the market value at the time the asset was purchased), then the difference between the market value and the book value reflects an investment gain or loss. Such a gain (or loss) is called an **unrealized gain (loss)**. Regulation to which the portfolio is subject may require an asset's value to be restated at market value, or **marked to market**, at certain intervals or in certain events (such as the sale of the asset). At that point, the

unrealized gains are said to be realized.¹² For illustration, assume an asset is purchased for \$100 and then appreciates to \$110. Prior to selling the asset, the \$10 gain is unrealized. Upon the sale of the asset, the unrealized gain converts to a realized gain.

Many tax jurisdictions distinguish between realized and unrealized gains and treat such gains differently in assessing taxes. Consequently, in taxable accounts, record keepers may report realized and unrealized gains and losses on account statements. For performance measurement, tax payments may constitute cash flows out of a portfolio and must be accounted for. The treatment and management of taxable portfolios can be complex because of the obvious dependence on tax laws and the status of unrealized versus realized gains.

2.1.7.4 Trade Date Accounting vs. Settlement Date Accounting Trade date accounting recognizes the value of assets and liabilities on the date of purchase or sale. **Settlement date accounting**, on the other hand, recognizes the value of assets or liabilities on the date when the actual exchange of cash and investments is completed. Because trade date accounting accurately captures the timing of the investment decision and reflects the correct economic value of the portfolio assets as of the transaction date, valuations as of the trade date are more appropriate for use in performance measurement. Further, the GIPS standards require that, for periods beginning on or after 1 January 2005, firms use trade date accounting.

2.2 Factors Affecting Returns and Related Adjustments

A number of costs can affect the actual increments to wealth, so different types of return can be defined. The following section introduces inflation.

2.2.1 Inflation: Nominal Returns vs. Real Returns

Economists distinguish between **nominal prices** (actual transaction prices, with no adjustment for inflation) and **real prices** (nominal prices adjusted for changes in purchasing power of currency units over time). The change in purchasing power is a measure of monetary inflation. In many countries, the price levels of standard baskets of goods are calculated and maintained by government agencies. A price index is a number calculated from such standardized baskets, from which a change in the price index can be interpreted as inflation. Examples of such price indexes are the consumer price index (CPI) in many countries and the Retail Prices Index (RPI) in the United Kingdom. Other examples of price indexes used to measure inflation are the Producer Price Index and the GDP deflator.

Market values of assets can be converted to alternate base currencies by means of spot exchange rates, but the “real” value of financial assets can only be defined with respect to a chosen price index. If I is the value of the consumer price index, then an **inflation rate**, IR , over a given period can be calculated as the percentage change in the index over the period. Using subscripts 0 and 1 to denote the beginning and the end of the period,

$$IR = \frac{I_1}{I_0} - 1$$

¹² An asset can also be marked to market for performance measurement purposes, which is a paper transaction that does not result in gains or losses being realized.

Given this measure of inflation, the “real value” of the currency alone has changed by IR . Therefore, the real value of a portfolio at the end of the period can be calculated by discounting the nominal value, V_1 , by the rate of inflation, or $V_1/(1 + IR)$. A **real return**, R^{real} , can thus be defined as

$$R^{\text{real}} = \frac{V_1/V_0}{I_1/I_0} - 1 = \frac{1 + R}{1 + IR} - 1 = \frac{R - IR}{1 + IR} \quad (10)$$

where R is the **nominal return** of the portfolio. For example, if R is 0.10 and IR is 0.04, then $R^{\text{real}} = (0.10 - 0.04)/1.04 = 0.0577$ or 5.77 percent. For small values of IR , the denominator adjustment in Equation 10 is small and the approximation $R^{\text{real}} = R - IR = 0.10 - 0.04 = 0.06$ or 6 percent is sometimes used. The corresponding expression for the nominal rate of return is

$$R = (1 + IR)(1 + R^{\text{real}}) - 1 \quad (11)$$

We emphasize here, however, that the real return in this equation is derived using the given price level index and cannot be measured independently.

Equation 11 can be interpreted further by multiplying out the right-hand side:

$$1 + R = 1 + IR + R^{\text{real}} + (IR \times R^{\text{real}})$$

The nominal return R is approximately the sum of the inflation rate IR and the real return R^{real} . The additional cross-product term $IR \times R^{\text{real}}$ represents the additional change from compounding. The formula for the real return R^{real} in Equation 10 as noted is called the **geometric difference** between the nominal return and the inflation rate. As seen, it is equal to the **arithmetic difference** $R - IR$ (sometimes used by itself as an approximation) divided by the inflation return relative $(1 + IR)$. Arithmetic and geometric differences arise naturally when portfolio returns are compared with other reference rates of return such as the inflation rate above or benchmark returns.

EXAMPLE 12

Nominal Return, Inflation Rate, and Real Return

Calculate the real return of a portfolio given the following nominal return and inflation rate.

Portfolio return	8.7 percent
Inflation rate	3.2 percent

Solution:

$$1.087/1.032 - 1 = 5.33\%.$$

2.2.2 Fees: Gross-of-Fees Returns vs. Net-of-Fees Returns

From the perspective of an investor, investment management fees are a cost of investing; therefore, a return that reflects the fees, or a **net-of-fee return**, is of value. From the perspective of a portfolio manager, however, some fees are not a direct consequence of portfolio decisions; portfolios managed identically might be offered at varying fee levels to different clients. Further, net-of-fee returns may not fairly reflect the skill of the manager when compared with a benchmark because benchmark returns typically are not calculated with a deduction for fees.

Consequently, portfolio managers and analysts tend to prefer **gross-of-fee returns** for analyzing the consequences of portfolio decisions. More specifically, gross-of-fee returns are the returns based on portfolio values reduced by any trading expenses incurred during the period but not reduced by investment management fees; net-of-fee returns are gross-of-fee returns reduced by investment management fees. Investment

management fees are typically asset based (a percentage of assets under management)¹³ or performance based (determined by the performance of the portfolio on an absolute basis or relative to a benchmark) but may take other forms as well.

Depending on the setting, fees may be incorporated into portfolio accounting in different ways. In regulated mutual funds, the **net asset value** (NAV) of the fund is based on valuing the portfolio net of any liabilities including fees and expenses which are payable. In such a setting, calculating performance from net asset values automatically gives a net-of-fee return. In institutional funds settings, it is more common for fees to be accrued per the investment management contract terms and paid at discrete dates. If the fee payment is treated as capital leaving the portfolio (i.e., as an external cash flow as discussed in Section 4), the calculated returns would be gross-of-fee returns. If, instead, the fee payment is treated as a simple reduction in the portfolio value, the calculated returns would be net-of-fee returns. Under the latter treatment, fees are typically accrued as a negative value between fee payment dates to ensure all intermediate period returns are calculated net-of-fees.¹⁴ Performance fees are also similarly accrued, normally based on the theoretical payment given the excess return achieved to date. If negative excess performance is suffered in subsequent periods, this may lead to the strange result in which the net-of-fee return is greater than the gross-of-fee return as the performance fee accrual is unwound.

As mentioned, either net-of-fee or gross-of-fee returns might be more readily calculated depending on the setting and the available portfolio accounting data. Given a net-of-fee return, an analyst may want to calculate a gross-of-fee return, or vice versa. Typically, approximations are used to calculate one from the other. To derive net-of-fee returns from gross-of-fee returns, one must first determine the type of investment management fee being charged and the frequency at which the fees are incurred. As an example, assume an annual asset-based fee of 0.50 percent (50 bps) is charged on a monthly basis (equal to $0.0050/12$ of the daily average of the value of assets under management during a month). To calculate an approximate net-of-fee return from a gross-of-fee return, the geometric difference¹⁵ between the gross-of-fee return and the fee percentage per month can be calculated. Similarly, to calculate an approximate gross-of-fee return from a net-of-fee return, the assumed fee percentage can be compounded with the net-of-fee return.

EXAMPLE 13

Gross-of-Fee and Net-of-Fee Returns

- 1 Calculate the net-of-fee return for Portfolio A and the gross-of-fee return for Portfolio B. Assume, as a simplification, that fees are accrued throughout the year and paid annually.

Portfolio A's gross-of-fee return for 2010: 13.5%; annual fee: 1.5%

Portfolio B's net-of-fee return for 2010: 10.4%; annual fee: 1.2%

- 2 Calculate the 2010 quarterly net-of-fee returns for Portfolio C. Assume, as a simplification, that fees are accrued and paid quarterly.

Portfolio C's annual fee: 1.2%, paid quarterly

Portfolio C's quarterly gross-of-fee returns for 2010:

1Q: 2.41%, 2Q: -1.68%, 3Q: 3.72%, 4Q: 0.28%

¹³ A fee calculated as a percentage of assets under management is called an *ad valorem* fee.

¹⁴ Another scenario is one where the fees are paid by the client from a separate account, not deducted from the portfolio. Under such a scenario the gross-of-fee return is calculated by ignoring the fee payment.

¹⁵ Calculating the geometric difference between two percentages involves adding 1 to each percentage, dividing the terms by each other, and then subtracting 1 from the result.

Solution to 1:

Portfolio A's net-of-fee return is

$$R = \frac{(1 + 13.5\%)}{(1 + 1.5\%)} - 1 = \frac{1.135}{1.015} - 1 = 11.8\%$$

Portfolio B's gross-of-fee return is

$$R = (1 + 10.4\%)(1 + 1.2\%) - 1 = 1.104 \times 1.012 - 1 = 11.7\%$$

Solution to 2:

Given the stated annual fee rate of 1.2 percent, the quarterly fee rate is $0.012/4 = 0.0030$. Portfolio C's net-of-fee returns are

$$1Q = \frac{1.0241}{1.0030} - 1 = 0.0210 = 2.10\%$$

$$2Q = \frac{0.9832}{1.0030} - 1 = -0.0197 = -1.97\%$$

$$3Q = \frac{1.0372}{1.0030} - 1 = 0.0341 = 3.41\%$$

$$4Q = \frac{1.0028}{1.0030} - 1 = -0.0002 = -0.02\%$$

2.2.3 Taxes: Pre-Tax Returns vs. Post-Tax Returns

Because of the nature of tax laws in many jurisdictions, calculating post-tax returns can be complicated. Because a tax liability is typically incurred when an asset is sold, assumptions about when assets are sold are critical. In a taxable portfolio, the decision to sell assets might be made either by the asset owner or by the portfolio manager. Therefore, a return that reflects the impact of taxes must be based on assumptions regarding the sale of assets and the tax rate at which the gains are taxed.

Mutual funds in the United States are required to report, or distribute, dividends and capital gains that are realized by the fund each year. Realized capital gains might have resulted from sell decisions made by the portfolio manager. Even if dividends and realized capital gains are reinvested in the portfolio, such distributions create a tax liability for the fund investors. Another type of tax liability occurs when an investor redeems shares of a mutual fund. In this event, unrealized capital gains are realized and become taxable. Obviously, post-tax returns must be accompanied with the assumptions by which they are calculated to enable correct interpretation of the returns.

EXAMPLE 14**Pre-Tax and Post-Tax Returns**

- 1 An asset was purchased for \$3,500,000 and then later sold for \$3,750,000. The applicable capital gains tax rate is 35 percent. Calculate the pre-tax and post-tax holding period returns.
- 2 For an initial investment of \$1,000 at the beginning of the year, a mutual fund reported dividend amounts of \$20 and distributed capital gains of \$34 for the year. If the account was liquidated at the end of the year, an additional capital gain of \$132 would be realized. Assume the following tax rates:

Tax rate on dividend income: 35%

Tax rate on capital gains: 15%

Calculate the pre-tax return, the post-tax return assuming the account is not liquidated at the end of the year, and the post-tax post-redemption return assuming the account is liquidated.

Solution to 1:

$$\text{Pre-tax return} = \frac{\$3,750,000}{\$3,500,000} - 1 = 0.0714 = 7.14\%$$

$$\text{Capital gain} = \$3,750,000 - \$3,500,000 = \$250,000$$

$$\text{Post-tax capital gain} = \$250,000(1 - 35\%) = \$162,500$$

$$\text{Post-tax return} = \frac{\$162,500}{\$3,500,000} = 0.0464 = 4.64\%$$

Solution to 2:

Gains after taxes can be calculated using the information in the table below:

	Pre-tax amount	Tax rate	Tax	Post-tax amount
Dividend income	\$20.00	35%	\$7.00	\$13.00
Capital gain distribution	\$34.00	15%	\$5.10	\$28.90
Post-redemption gain	\$132.00	15%	\$19.80	\$112.20

$$\text{Pre-tax return} = \frac{\$20 + \$34 + \$132}{\$1,000} = 0.1860 = 18.60\%$$

$$\begin{aligned} &\text{Post-tax pre-redemption return} \\ &= \frac{\$13.00 + \$28.90 + \$132.00}{\$1,000} = 0.1739 = 17.39\% \end{aligned}$$

$$\begin{aligned} &\text{Post-tax post-redemption return} \\ &= \frac{\$13.00 + \$28.90 + \$112.20}{\$1,000} = 0.1541 = 15.41\% \end{aligned}$$

MECHANICS OF MULTI-PERIOD RETURNS

3

In principle, a return over a long holding period (a year, for instance) can be calculated in the same manner as that for a shorter holding period (such as a month) using beginning and ending market values—if there are no external cash flows within the period. It is common, however, to calculate returns over short holding periods and then to combine or link returns over multiple periods. Such a procedure is necessary when external cash flows exist, as we explain in Section 4. Storing return (or return relative) data allows users the flexibility of calculating returns over arbitrary holding periods.

3.1 Compounding Rates of Return

If returns are generated at a constant rate over time, the portfolio's value will grow faster than it will if returns are generated in proportion to time. Such growth is called geometric growth, as opposed to linear growth. This fact is the result of **compounding**. A compounded return over two periods can be calculated as

$$(1 + R_1)(1 + R_2) - 1 = R_1 + R_2 + R_1R_2$$

where the subscripts indicate time periods. The term $R_1 + R_2$ reflects linear growth in value over time, while the interaction term R_1R_2 represents “interest on interest,” the key to geometric growth, also known as compound growth.

3.1.1 Chain Linking Returns

Suppose a portfolio is valued at month ends over the course of a year. Let V_t be the portfolio value at the end of month t , and let V_0 be the portfolio value at the beginning of the year. If there were no external cash flows into or out of the portfolio during the year, then the return relative for the year can be written as the product of the monthly return relatives,

$$\frac{V_{12}}{V_0} = \frac{V_1}{V_0} \times \frac{V_2}{V_1} \times \dots \times \frac{V_{12}}{V_{11}} \quad (12)$$

because the intermediate values V_1, V_2, \dots, V_{11} cancel in the expression on the right-hand side. Substituting the monthly holding period returns $V_t/V_{t-1} = 1 + R_t$ into the preceding expression leaves

$$\frac{V_{12}}{V_0} = (1 + R_1)(1 + R_2) \cdots (1 + R_{12})$$

More generally, if $R = V_T/V_0 - 1$ is the return over T periods, then R can be calculated from the **linking formula**:

$$R = \prod_{t=1}^T (1 + R_t) - 1 \quad (13)$$

where the product operator $\Pi(\bullet)$ denotes the product of terms. The procedure described by this formula is also called **chain linking** returns or **geometric compounding** of returns. When there are no external cash flows, the cumulative return R can be interpreted as the change in value of the portfolio as a fraction of the beginning value.

EXAMPLE 15

Chain Linking Returns

Calculate the cumulative return for the following 5 years:

Year 1	10.5%
Year 2	-3.6%
Year 3	20.7%
Year 4	6.4%
Year 5	12.3%

Solution:

$$1.105 \times 0.964 \times 1.207 \times 1.064 \times 1.123 - 1 = 53.63\%$$

Equation 13 applies to compounding **discrete returns** (returns measured over finite time intervals). The compounding of discrete returns is called **discrete compounding**. It is also possible to define a continuously compounded return: the return which results when a finite time interval is divided into infinitely many parts. The compounding of continuously compounded returns is called **continuous compounding**. The next section explains these concepts in detail.

3.1.2 Discrete and Continuous Compounding

Whenever an interest rate is quoted, the compounding frequency must be specified. By convention, the **stated annual rate** of interest is the periodic rate multiplied by the number n of compounding periods per year. Thus, the periodic rate associated with a stated annual rate i is i/n . The **effective annual rate** is defined as¹⁶

$$\text{EAR} = \left(1 + \frac{i}{n}\right)^n - 1 \quad (14)$$

Thus, if the stated annual rate is 12 percent and there are 12 compounding periods per year, the periodic rate is $12\%/12 = 1\%$ and the $\text{EAR} = (1.01)^{12} - 1 = 0.1268$ or 12.68 percent. In other words, a dollar invested at a 12 percent stated annual rate with monthly compounding produces the same amount in one year as one dollar invested at 12.68 percent without compounding.

In principle, the compounding frequency can be increased without bound, resulting in continuous compounding. For a given nominal rate, as the compounding frequency gets large, the annual return approaches a limit

$$\lim_{n \rightarrow \infty} \left(1 + \frac{i}{n}\right)^n = e^i$$

where e is the base of the natural logarithm, a constant approximately equal to 2.718. Also, one dollar invested at an annual rate R without compounding gives the same ending amount as $i_c = \ln(1 + R)$ compounded continuously. Consequently, i_c is called a continuously compounded return. For example, Exhibit 3 shows the nominal rates at different compounding frequencies that give the same effective annual rate. The EAR associated with i_c compounded continuously is $\exp(i_c) - 1$. Thus, 4 percent continuously compounded is equal to $e^{0.04} - 1 = 0.0408$ or 4.08 percent.

Exhibit 3 Equivalent Effective Annual Rates of Return

Compounding Frequency	Stated Annual Rate (%)
Annual	12.0000
Semi-annual	11.6601
Quarterly	11.4949
Monthly	11.3866
Weekly	11.3452
Daily (365 days)	11.3346
Continuously compounded	11.3329

¹⁶ The terminology for the concepts introduced here follows CFA® curriculum usage; a wide variety of synonyms may be encountered in various world markets.

Although the interpretation of continuous compounding justifies the terminology, the use of logarithms can also be thought of as simply a transformation. A useful property of logarithms is the fact that the logarithm of a product is equal to the sum of the logarithms of individual terms. For example, the linking formula, Equation 13, can be written as

$$R = \exp\left(\sum_{t=1}^T \ln(1 + R_t)\right) - 1$$

where $\exp(\bullet)$ is the exponential function. Computationally, it is sometimes convenient to calculate sums of terms instead of products.

EXAMPLE 16

Continuously Compounded Returns

- 1 What is the continuously compounded rate that produces the same cumulative return as in the solution to Example 15?
 - A 42.94%
 - B 47.28%
 - C 53.63%
- 2 Given a stated interest rate of 10.9 percent paid semi-annually, calculate the effective annual rate.
 - A 11.0%
 - B 11.1%
 - C 11.2%

Solution to 1:

A is correct. $\ln(1 + 53.63\%) = 42.94\%$

Solution to 2:

C is correct.

$$R = \left(1 + \frac{i}{n}\right)^n - 1 = \left(1 + \frac{0.109}{2}\right)^2 - 1 = 1.1120 - 1 = 11.2\%$$

3.1.3 Practical Issues

Although some investments (such as term deposits) have well-defined compounding periods, in performance measurement the holding period of a return is generally dictated by the availability of valuation data. The periodicity of the return is not a characteristic of the investment; it is a parameter chosen by the performance measurer. In principle, market values may exist at any point in time regardless of whether they are measured and recorded. For analytical purposes, discrete compounding and continuous compounding constitute alternative models of growth. The models are equivalent if portfolio values match at the given valuation time points. The choice of holding period or the use of continuous compounding depends on the nature of the analyst's task at hand and the data with which the analyst is working.

EXAMPLE 17**Equivalent Returns with Different Compounding Intervals**

Calculate the effective annual rate given the stated annual rate and number of compounding periods for the three cases given below.

Stated Annual Rate	Number of Compounding Periods per Year	Effective Annual Rate?
12%	4	A
10%	12	B
8%	continuous	C

Solution to A:

The periodic rate of return is $12\%/4 = 3\%$. $\text{EAR} = (1.03)^4 - 1 = 0.1255$ or 12.55%.

Solution to B:

The periodic rate of return is $10\%/12 = 0.8333\%$. $\text{EAR} = (1.008333)^{12} - 1 = 0.1047$ or 10.47%.

Solution to C:

$e^{0.08} - 1 = 0.0833$ or 8.33 percent.

3.2 Averaging Rates of Return

Returns calculated over multiple holding periods can be summarized by averaging over time. Two methods of averaging, arithmetic and geometric, give different results and are useful for different purposes.

3.2.1 Arithmetic Mean Rate of Return

Given a collection R_t ($t = 1, \dots, T$) of returns over time, the **arithmetic mean rate of return** is calculated as

$$\bar{R} = \frac{1}{T} \sum_{t=1}^T R_t \quad (15)$$

The arithmetic mean has well-known statistical properties, which make the calculation important for many analytical purposes. We address this point further in Section 3.2.3.

If the holding period of the return is $1/n$ years (where $n = 12$ corresponds to monthly, for example), then T periods span T/n years. The **annualized arithmetic mean rate of return** is the arithmetic mean multiplied by n :

$$\bar{R}^{ann} = n\bar{R} = \frac{n}{T} \sum_{t=1}^T R_t \quad (16)$$

3.2.2 Geometric Mean Rate of Return

The **geometric mean rate of return** is defined by the formula

$$\bar{R}_G = \left(\prod_{t=1}^T (1 + R_t) \right)^{1/T} - 1 \quad (17)$$

The geometric mean return can be interpreted as the constant periodic rate of return that results in the same ending value created by the given returns if there are no external cash flows. This fact is seen by linking the returns on both sides of the equation:

$$(1 + \bar{R}_G)^T = \prod_{t=1}^T (1 + R_t) = \frac{V_T}{V_0}$$

The **annualized geometric mean rate of return** is the constant rate, assuming annual compounding, that results in the same ending value. Assuming the periodic returns consist of n periods per year, T periods span T/n years. The annualized geometric mean return can be calculated as

$$\bar{R}_G^{ann} = (1 + \bar{R}_G)^n - 1 = \left(\prod_{t=1}^T (1 + R_t) \right)^{n/T} - 1 \quad (18)$$

The annualized geometric mean return is also called compound annual return or simply “annualized return.”

EXAMPLE 18

Arithmetic and Geometric Mean Rates of Return, Annualized

Calculate arithmetic and geometric mean rates of return using the returns below (repeated from Example 15):

Year 1	10.5%
Year 2	-3.6%
Year 3	20.7%
Year 4	6.4%
Year 5	12.3%

Solution:

$$\text{Arithmetic mean} = \frac{10.5\% - 3.6\% + 20.7\% + 6.4\% + 12.3\%}{5} = 9.26\%$$

$$\text{Geometric mean} = (1.105 \times 0.964 \times 1.207 \times 1.064 \times 1.123)^{1/5} - 1 = 8.967\%$$

Given the different ways of averaging (arithmetic and geometric), users of return data should take note of what a data provider or performance reporter means by “mean return.” When presenting historical portfolio returns, the geometric, rather than arithmetic, mean return must be used. The geometric mean is more accurate when measuring portfolio returns. It takes into account the compounding interaction between individual returns in addition to the returns themselves. When used in the context of investment returns, the arithmetic mean can be misleading concerning the effects of a series of returns on final wealth, as illustrated in Example 19.

EXAMPLE 19**Arithmetic and Geometric Mean Rates of Return, Annualized**

Calculate arithmetic and geometric mean rates of return using the returns for the following five years, and then determine which measure is more appropriate:

Year 1	100%
Year 2	-25%
Year 3	0%
Year 4	50%
Year 5	-75%

Solution:

$$\text{Arithmetic mean} = \frac{100\% - 25\% + 0\% + 50\% - 75\%}{5} = 10.00\%$$

$$\text{Geometric mean} = (2.000 \times 0.750 \times 1.000 \times 1.500 \times 0.250)^{1/5} - 1 = -10.87\%$$

The geometric mean is more appropriate. The positive 10 percent arithmetic mean return implies that the portfolio has added value on an average annual basis. However, the compound growth rate for the five-year period is -43.75 percent, which indicates the opposite. The geometric mean more accurately reflects reality.

Although the formulas for mean returns can be annualized, caution should be used if the number T of returns is less than the number n of periods per year. In this case, the returns span a period less than a year and annualizing may be misleading. Annualizing for periods less than one year results in simulated or hypothetical performance and is thus considered a poor performance measurement practice.

EXAMPLE 20**Annualizing Partial-Year Returns**

A portfolio achieved a return of 12 percent during first quarter 2011 and a return of 8 percent during second quarter 2011. Calculate the annualized rate of return for the full year 2011 and explain why the result is or is not appropriate.

Solution:

Annualized return would be $(1.12 \times 1.08)^{2/1} - 1 = 46.31\%$, but annualizing partial-year returns is not appropriate because doing so assumes that the performance results achieved during the partial year will continue at the same pace for the remainder of the year. The annualized results would be hypothetical and potentially misleading and, therefore, should not be included in performance presentation materials.

3.2.3 Observations about the Arithmetic and Geometric Means¹⁷

One property of arithmetic and geometric mean returns is notable: The geometric mean return is less than or equal to the arithmetic mean return.¹⁸

$$\bar{R}_G \leq \bar{R}$$

A good approximation of the difference can be calculated from

$$\bar{R}_G \cong \bar{R} - \frac{1}{2}s^2 \quad (19)$$

where s is the sample standard deviation of the returns R_t :

$$s^2 = \frac{1}{T-1} \sum_{t=1}^T (R_t - \bar{R})^2$$

Thus, if the arithmetic mean return is 10 percent and the standard deviation of returns is 0.20, according to Equation 19 the geometric mean return is approximately $0.10 - \frac{1}{2}(0.20)^2 = 0.08$ or 8 percent.

Arithmetic and geometric mean returns are useful for different purposes. If we need to estimate an expected or average return over a one-period horizon of stated length, a historical arithmetic mean return is an appropriate estimator. For statistical analysis, when a model is fitted with parameters estimated from data, the arithmetic mean normally is the appropriate estimator. The geometric mean is affected by volatility, whereas the arithmetic mean is not. In contrast, for summarizing the effect on wealth of observed investment returns, the geometric mean return is the only return to use: It represents the compound rate of growth of an investment—that is, the rate of return that if earned each period on the initial investment would match the actual cumulative return achieved.

EXAMPLE 21

Arithmetic and Geometric Mean Rates of Return, Volatility Dependence

Exhibit 4 gives quarterly returns for the Russell 3000 Index, a broad market index of US equities, and for three-month Libor, a rate commonly used as a proxy for the risk-free rate. Calculate arithmetic and geometric mean quarterly rates of return using the returns for the three-year period presented.

Exhibit 4 Quarterly Total Returns of Two Indexes (Percent)

Quarter Ending	3-Month Libor USD	Russell 3000 Index
31-Mar-2006	1.16	5.34
30-Jun-2006	1.27	-1.94
30-Sep-2006	1.33	4.61
31-Dec-2006	1.31	7.09

¹⁷ For further reading on this subject, consult Christopherson, Cariño, and Ferson (2009).

¹⁸ This fact can be shown by using logarithms:

$$\ln(1 + \bar{R}_G) = \ln \left[\left(\prod_{t=1}^T (1 + R_t) \right)^{1/T} \right] = \frac{1}{T} \sum_{t=1}^T \ln(1 + R_t) \leq \ln \left(\frac{1}{T} \sum_{t=1}^T (1 + R_t) \right) = \ln(1 + \bar{R})$$

where the inequality follows from concavity of the logarithm function. Strict inequality holds if not all the R_t are equal. The amount by which the geometric average is less depends on the volatility: The greater the volatility of the returns, the greater the difference between the two averages.

Exhibit 4 (Continued)

Quarter Ending	3-Month Libor USD	Russell 3000 Index
31-Mar-2007	1.31	1.24
30-Jun-2007	1.31	5.95
30-Sep-2007	1.33	1.53
31-Dec-2007	1.27	-3.35
31-Mar-2008	0.81	-9.50
30-Jun-2008	0.68	-1.62
30-Sep-2008	0.72	-8.58
31-Dec-2008	0.69	-22.74

Solution:

The quarterly arithmetic means are as follows:

Three-month Libor USD:

$$\frac{(1.16 + 1.27 + 1.33 + 1.31 + 1.31 + 1.31 + 1.33 + 1.27 + 0.81 + 0.68 + 0.72 + 0.69)}{12}$$

$$= 1.10\%$$

Russell 3000 Index:

$$\frac{(5.34 - 1.94 + 4.61 + 7.09 + 1.24 + 5.95 + 1.53 - 3.35 - 9.5 - 1.62 - 8.58 - 22.74)}{12}$$

$$= -1.83\%$$

The quarterly geometric means are:

Three-month Libor USD:

$$(1.0116 \times 1.0127 \times 1.0133 \times 1.0131 \times 1.0131 \times 1.0131 \times 1.0133 \times 1.0127 \times 1.0081 \times 1.0068 \times 1.0072 \times 1.0069)^{1/12} - 1 = 1.10\%$$

Russell 3000 Index:

$$(1.0534 \times 0.9806 \times 1.0461 \times 1.0709 \times 1.0124 \times 1.0595 \times 1.0153 \times 0.9665 \times 0.905 \times 0.9838 \times 0.9142 \times 0.7726)^{1/12} - 1 = -2.20\%$$

Equation 19 provides a good approximation to the quarterly geometric mean for the Russell 3000 Index. The sample standard deviation of returns or s is 8.52 percent.

$$\bar{R}_G \cong -1.83\% - \frac{8.52\%^2}{2} = -1.83\% - 0.36\% = -2.19\%$$

The quarterly returns for the three-month Libor USD are relatively consistent during the three-year period, and because of the low volatility, the arithmetic and geometric returns are identical when rounded to two decimal places. The same cannot be said for the Russell 3000. The quarterly returns for this index vary significantly throughout the period under review, leading to considerable disparity between the arithmetic and geometric means. When evaluating more volatile assets or time periods, the differences between the arithmetic and geometric means will be even more extreme.

Geometric mean returns are useful for accurately summarizing the ending wealth that has been produced by an investment. From the definition in Section 3.2.2, we see that the portfolio ending value can be calculated from the beginning value and the

geometric mean return—there is a one-to-one relationship between the geometric mean return and the relative value. In contrast, ending wealth cannot be recovered from the arithmetic mean return. In general, therefore, if the intent is to accurately summarize and report *past* returns, as is the case in performance evaluation, the geometric mean return must be used. If the intent is to forecast *future* returns or infer parameters of statistical models, the arithmetic mean return is more useful.

3.2.4 Arithmetic and Geometric Excess Returns

The difference between a portfolio return R and a benchmark return B is called the **excess return**. Excess return can be defined arithmetically or geometrically:

$$\text{Arithmetic excess return} = R - B$$

$$\text{Geometric excess return } G = \frac{1 + R}{1 + B} - 1 = \frac{R - B}{1 + B}$$

Note that the geometric excess return is simply the arithmetic excess return divided by the wealth ratio of the benchmark.

The arithmetic excess return can be thought of as the profit in excess of a notional or benchmark fund expressed as a percentage of the initial amount invested. The geometric excess return can be thought of as the exact same profit in excess of the notional or benchmark fund, but it is expressed as a percentage of the *ending* value of the notional or benchmark fund.

EXAMPLE 22

Arithmetic and Geometric Excess Returns

Calculate both the geometric and arithmetic excess returns of the following portfolio for the month of May:

Portfolio 30 April	\$1,000,000
Portfolio 31 May	\$1,120,000
Benchmark return for May	10%

Solution:

$$\text{Portfolio return} = \frac{\$1,120,000 - \$1,000,000}{\$1,000,000} = 12\%$$

$$\text{Arithmetic excess return} = 12\% - 10\% = 2\%$$

$$\text{Geometric excess return} = \frac{1.12}{1.10} - 1 = 1.82\%$$

Note that if the portfolio had been invested in the benchmark, the notional value on 31 May would have been \$1,100,000. Thus the value added relative to investing in the benchmark was \$1,120,000 - \$1,100,000 = \$20,000. This, in turn, equates to an

$$\text{Arithmetic excess return} = \frac{\$20,000}{\$1,000,000} = 2\%$$

$$\text{and a Geometric excess return} = \frac{\$20,000}{\$1,100,000} = 1.82\%$$

For many practitioners, the simple subtraction between portfolio return and benchmark return or arithmetic excess return is the natural and intuitive presentation of excess return and is by far the most common method used worldwide. For other practitioners, the less intuitive geometric excess return is preferred because the returns are compoundable over time, are convertible across currencies, and are proportionate.

Geometric excess returns are compoundable because both the cumulative portfolio and benchmark returns are the compound of each sub-period, as follows:

$$1 + R = (1 + R_1) \times (1 + R_2) \times \dots \times (1 + R_T)$$

and

$$1 + B = (1 + B_1) \times (1 + B_2) \times \dots \times (1 + B_T)$$

Therefore, the cumulative geometric excess return is simply the compound of each sub-period's geometric excess return:

$$1 + G = \frac{1 + R}{1 + B} = \frac{(1 + R_1)}{(1 + B_1)} \times \frac{(1 + R_2)}{(1 + B_2)} \times \dots \times \frac{(1 + R_T)}{(1 + B_T)}$$

$$1 + G = \frac{1 + R}{1 + B} = (1 + G_1) \times (1 + G_2) \times \dots \times (1 + G_T)$$

No such relationship exists for arithmetic excess returns. They neither sum nor compound to the cumulative period excess returns.

EXAMPLE 23

Cumulative Arithmetic and Geometric Excess Returns

Calculate annual arithmetic and geometric excess returns for the portfolio data shown in the following table:

	1st Quarter	2nd Quarter	3rd Quarter	4th Quarter
Portfolio return	7.1%	11.5%	-5.0%	8.5%
Benchmark return	5.3%	14.1%	-6.2%	4.7%
Arithmetic excess	1.8%	-2.6%	1.2%	3.8%
Geometric excess	1.71%	-2.28%	1.28%	3.63%

Solution:

The annual portfolio and benchmark returns are

$$\text{Portfolio: } (1 + 7.1\%) \times (1 + 11.5\%) \times (1 - 5.0\%) \times (1 + 8.5\%) - 1 = 23.09\%$$

$$\text{Benchmark: } (1 + 5.3\%) \times (1 + 14.1\%) \times (1 - 6.2\%) \times (1 + 3.8\%) - 1 = 17.99\%$$

The annual arithmetic and geometric excess returns are thus

$$\text{Arithmetic: } 23.09\% - 17.99\% = 5.09\%$$

$$\text{Geometric: } \frac{(1 + 23.09\%)}{(1 + 17.99\%)} - 1 = 4.32\%$$

The quarterly geometric excess returns compound to the same annual geometric excess return, as follows:

$$(1 + 1.71\%) \times (1 - 2.28\%) \times (1 + 1.28\%) \times (1 + 3.63\%) - 1 = 4.32\%$$

The quarterly arithmetic excess returns do not link directly to the annual arithmetic excess return.

Because geometric excess returns are compoundable, the annualized geometric excess return can be calculated in the same way as the annualized return using Equation 18. The annualized arithmetic excess return cannot be derived directly from the arithmetic cumulative return but can be derived from the annualized portfolio and benchmark returns.

EXAMPLE 24**Annualized Arithmetic and Geometric Excess Returns**

Calculate annualized arithmetic and geometric excess returns for the three-year portfolio data shown in the following table:

	Year 1	Year 2	Year 3	Three-Year Cumulative
Portfolio return	17.6%	-5.7%	12.4%	24.65%
Benchmark return	15.8%	-7.6%	14.9%	22.94%
Arithmetic excess	1.8%	1.9%	-2.5%	1.71%
Geometric excess	1.55%	2.06%	-2.18%	1.39%

Solution:

Annualized portfolio return: $(1 + 24.65\%)^{1/3} - 1 = 7.62\%$

Annualized benchmark return: $(1 + 22.94\%)^{1/3} - 1 = 7.13\%$

Annualized geometric excess return: $(1 + 1.39\%)^{1/3} - 1 = 0.46\%$

or Annualized geometric excess return: $\frac{(1 + 7.62\%)}{(1 + 7.13\%)} - 1 = 0.46\%$

The annualized arithmetic excess return is more problematic and, by convention, is normally calculated as the arithmetic difference of the annualized portfolio and annualized benchmark returns:

Annualized arithmetic excess return: $7.62\% - 7.13\% = 0.49\%$

This result cannot be derived directly from the arithmetic difference of the three-year cumulative return, 1.71 percent.

Geometric excess returns are convertible across currencies. Using Equation 8a applied to both portfolio and benchmark returns, we observe the following relationship:

$$\frac{1 + R}{1 + B} = \frac{(1 + R^{FC}) \times (1 + R^C)}{(1 + B^{FC}) \times (1 + R^C)} = \frac{1 + R^{FC}}{1 + B^{FC}}$$

Note that the currency return in the benchmark is the same as the currency in the portfolio. This relationship means the geometric excess return will be the same in whatever base currency the portfolio and benchmark returns are presented. The arithmetic excess return will differ for each currency (provided there is at least some element of currency return). Clearly you cannot add different value simply by expressing the same portfolio and benchmark returns in different base currencies.

Currency conversions are particularly relevant for portfolio managers wishing to present portfolio performance to a client with a different domestic currency. Both portfolio and benchmark returns will be converted to the desired base currency return using the same exchange rates.

EXAMPLE 25**Geometric Excess Returns and Currency Conversions**

Convert the portfolio data shown in the table below from US dollars into euros assuming the US dollar appreciated against the euro by 8 percent during the period.

Portfolio return	15.0%
Benchmark return	10.0%
Arithmetic excess return	5.0%
Geometric excess return	4.55%

Solution:

Portfolio return in euros: $(1 + 15\%) \times (1 + 8\%) - 1 = 24.2\%$

Benchmark return in euros: $(1 + 10\%) \times (1 + 8\%) - 1 = 18.80\%$

Arithmetic excess return in US dollars: $15\% - 10\% = 5\%$

Geometric excess return in US dollars: $\frac{(1 + 15\%)}{(1 + 10\%)} - 1 = 4.55\%$

Arithmetic excess return in euros: $24.2\% - 18.8\% = 5.4\%$

Geometric excess return in euros: $\frac{(1 + 24.2\%)}{(1 + 18.8\%)} - 1 = 4.55\%$

Note that the geometric excess return is the same in both currencies, whereas the arithmetic excess return appears to change.

Geometric excess returns are proportionate because they represent the added value relative to the growth in the benchmark. To illustrate this, take the example of two portfolio managers, one delivering 51 percent performance against a benchmark return of 50 percent and the other delivering 11 percent performance against a benchmark of 10 percent. The arithmetic excess returns of both managers are equivalent at 1 percent, but intuitively, the manager that has grown the portfolio by 11 percent against an opportunity of 10 percent has done better than the manager that has grown the portfolio by 51 percent against the opportunity of 50 percent. Geometrically, we capture that outperformance in that 11 percent versus 10 percent represents a $(1.11/1.10) - 1 = 0.91\%$ excess return, whereas 51 percent versus 50 percent represents only a $(1.51/1.50) - 1 = 0.67\%$ excess return.

RATES OF RETURN WITH EXTERNAL CASH FLOWS**4**

We describe measures of return with external cash flows in this section. **External cash flows** are generally defined as capital (cash or investments) that enters or exits a portfolio. External cash flows occur, for example, as contributions by the plan sponsor or payments out of the portfolio to pension plan beneficiaries or as deposits and withdrawals by investors, in the case of a mutual fund or a personal portfolio.

There is an exception to the definition of external cash flow as capital that enters or exits the portfolio. If the intent is to calculate a return that reflects the impact of fees and expenses, which might literally be paid by transferring capital out of the portfolio, then such payments can be treated as reductions in the portfolio value. In

this case, such payments would not be considered external cash flows. The calculation of net asset values (NAVs) of mutual funds is an example of such a treatment. Returns calculated from portfolio values recorded this way are considered net-of-fee returns. If, instead, fee payments are regarded as external cash flows and treated using the methods in this section, the resulting returns are gross-of-fee returns.

These external cash flows complicate the calculation of returns. For example, consider a bank account that began the year with a balance of \$100. Suppose we deposited an additional \$100 into the account near the end of the year. If the account pays 2 percent interest per year, we might have about \$202 in the account at the end of the year. A calculation from beginning and ending values, $(202 - 100)/100 = 102\%$, would not give a meaningful measure of the interest paid by the bank.

Key issues for performance measurement raised by external cash flows are as follows:

- The external cash flows might not be the result of investment decisions by the portfolio manager. A calculated return that is influenced by external cash flows might not fairly depict the value of the manager's decisions.
- Returns cannot be calculated only from beginning and ending values. Valuations at other times may also be required, as well as information on the magnitude and timing of external cash flows.
- Valuations at the required dates and accurate records of the amounts and timing of cash flows might be expensive to obtain or simply unavailable. In this case, an approximate method of calculating return must be used.

This section describes measures of return with external cash flows and standard approximations for cases where complete valuation data are lacking.

4.1 Money-Weighted Rate of Return

The first return concept that we discuss here is the **internal rate of return** (IRR) calculation. Loosely speaking, the IRR is the constant rate of return that would produce the total ending value of a portfolio given the magnitude and timing of the cash flows. It gives greater weight to time periods where the portfolio has greater value than to periods where the portfolio has less value. For this reason, the IRR is also known as a **money-weighted rate of return**.

The IRR is calculated from the beginning value V_0 and ending value V_1 of the portfolio and from the amount and timing of the external cash flows. By convention, a cash flow into the portfolio is a positive flow. If a cash flow C occurred midway through a holding period, then the beginning and ending values would be related by

$$V_1 = V_0(1 + R) + C(1 + R)^{1/2}$$

If there were multiple cash flows C_k ($k = 1, 2, \dots, K$), this expression generalizes to

$$V_1 = V_0(1 + R) + \sum_{k=1}^K C_k(1 + R)^{W_k} \quad (20)$$

where W_k is the fraction of the period over which cash flow k applies, namely,

$$W_k = \frac{\text{TD} - D_k}{\text{TD}} \quad (21)$$

where TD is the total number of calendar days in the period and D_k is the number of days since the beginning of the period. This **day-weighting fraction** assumes end-of-day cash flows. For beginning-of-day cash flows or other timing assumptions, the fraction should be adjusted accordingly. For example, assuming beginning-of-day cash flows, we would use $\text{TD} - D_k + 1$ in the numerator.

The IRR is the rate R that satisfies Equation 20. Unfortunately, there is no closed-form formula for the IRR and it must be calculated by a numerical approximation algorithm. This fact is rarely a problem because many spreadsheet programs and financial calculators have IRR functions. A possible problem arises, however, because for some patterns of cash flows, there are multiple values of R that satisfy Equation 20. For cash flows that are small relative to the beginning and ending values, however, an IRR can usually be found.

Notice that the IRR does not require valuations at the times of cash flows, only at the beginning and the end of a period. Of course, if there are no cash flows over a period, then the IRR gives exactly the same result as the familiar holding period return.

EXAMPLE 26

Internal Rate of Return: Calculation

Calculate the internal rate of return of a portfolio with the following valuation points and external cash flow (where dollar amounts are in millions):

Market value	31 March	\$56.3
Market value (prior to cash flow)	11 April	\$58.2
Contribution (made at end of the day)	11 April	\$9.8
Market value	30 April	\$69.6

Solution:

$$69.6 = 56.3 \times (1 + R) + 9.8 \times (1 + R)^{19/30}. R = 5.61\% \text{ satisfies the above equation: } 56.3 \times 1.0561 + 9.8 \times (1.0561)^{19/30} = 69.6.$$

Note: The mid-period valuation on 11 April is not necessary for completing the IRR calculation.

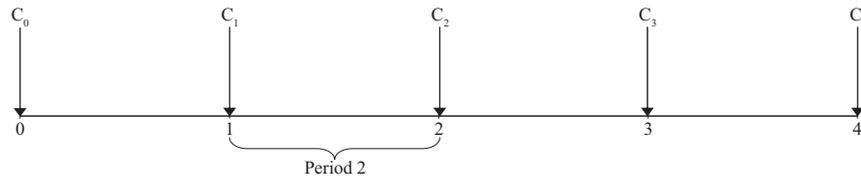
4.2 Time-Weighted Rate of Return

The decision to make an external cash flow is normally not under the control of the portfolio manager. As we have noted, those decisions may significantly influence the money-weighted rate of return. A general principle of evaluation, however, is that a person or entity should be judged only on the basis of their own actions or actions under their control. To isolate the effects of the portfolio manager's decisions, the measure of return should be insensitive to external cash flows. Such a return measure is the time-weighted rate of return (TWRR).

In the investment management industry, the time-weighted rate of return is the preferred performance measure. The TWRR measures the compound rate of growth of \$1 initially invested in the portfolio over a stated measurement period. In contrast to the money-weighted rate of return, the time-weighted rate of return is not affected by cash withdrawals or additions to the portfolio. As noted earlier, the money-weighted return gives greater weight to time periods where the portfolio has greater value than to periods where the portfolio has less value. For example, if a client gives an investment manager more funds to invest at an unfavorable time, the money-weighted rate of return will tend to be depressed. If a client adds funds at a favorable time, the money-weighted return will tend to be elevated. The time-weighted rate of return removes these effects. The term "time-weighted" refers to the fact that returns are averaged over time. The key idea is to distinguish the growth in value due to the investments held from the growth in value due to external cash flows. To do so, the portfolio values *at the times of the cash flows* are required, in addition to the beginning and ending values.

Consider a given holding period within which external cash flows have occurred. Divide the period into sub-periods corresponding to the times of the cash flows, and let C_t be the cash flow at time t , as illustrated in Exhibit 5.

Exhibit 5 Times of Cash Flows



Let V_t be the market value of the portfolio *immediately before the cash flow* at time t , and let V'_t be the market value of the portfolio *immediately after the cash flow* at time t . These values are related by

$$V'_t = V_t + C_t$$

For completeness, let V_0 be the overall beginning value and let V_T be the overall ending value of the portfolio (before cash flow at time T). The time-weighted rate of return R for the full period is defined by

$$R = \left(\frac{V_1}{V_0} \times \frac{V_2}{V'_1} \times \frac{V_3}{V'_2} \times \frac{V_4}{V'_3} \times \dots \times \frac{V_T}{V'_{T-1}} \right) - 1 \quad (22)$$

Note that compared with Equation 12 in Section 3.1.1, the value V_t just before the cash flow at time t need not equal the value V'_t just after the cash flow at time t and, therefore, does not cancel in the preceding expression. This arises on the days when there is a cash flow, as we see in Example 27 below. It is natural, however, to define a holding period return as $R_t = V_{t+1}/V'_t - 1$ and to calculate the return over T periods by linking:

$$R = (1 + R_1)(1 + R_2) \dots (1 + R_T) - 1 \quad (23)$$

EXAMPLE 27

Time-Weighted Rate of Return: Calculation

- 1 Calculate the time-weighted rate of return of a portfolio with the following valuation points and external cash flow:

Market value	31 March	\$56.3 million
Market value (prior to cash flow)	11 April	\$58.2 million
Contribution (made at end of the day)	11 April	\$9.8 million
Market value	30 April	\$69.6 million

- 2 Calculate the time-weighted rate of return for the following portfolio:

Date (t)	Market Value (V_t) Prior to External Cash Flows	Transaction
30 April	\$73.7 million	
12 May	\$69.3 million	\$15.3 million contribution
17 May	\$87.3 million	\$2.7 million dividend reinvested

Date (t)	Market Value (V_t) Prior to External Cash Flows	Transaction
26 May	\$89.7 million	\$8.1 million withdrawal
31 May	\$84.7 million	

Solution to 1:

$$R = \frac{\$58.2}{\$56.3} \times \frac{\$69.6}{\$58.2 + \$9.8} - 1 = 1.03375 \times 1.02353 - 1 = 0.0581 = 5.81\%$$

Compare this result with the IRR calculated in Example 26.

Solution to 2:

$$\begin{aligned} R &= \frac{\$69.3}{\$73.7} \times \frac{\$87.3}{\$69.3 + \$15.3} \times \frac{\$89.7}{\$87.3} \times \frac{\$84.7}{\$89.7 - \$8.1} - 1 \\ &= 0.9403 \times 1.0319 \times 1.0275 \times 1.038 - 1 \\ &= 0.0349 = 3.49\% \end{aligned}$$

Note that it is not necessary for mathematical accuracy in calculating time-weighted rate of return to revalue the portfolio at the time a dividend is reinvested. In the solution above, the second and third factors can be combined as $89.7/(69.3 + 15.3)$, bypassing the valuation of the portfolio on May 17.

The time-weighted return has a natural interpretation in terms of the closely related concept of **unit value pricing** of mutual funds or other funds in which the assets of multiple investors are commingled (i.e., combined). In such a fund, investors may purchase or sell shares or units of the fund. The net asset value (NAV) of a unit of the fund is the total market value V of the portfolio (after netting liabilities) divided by the number of units U existing at the time of the valuation:

$$\text{NAV} = \frac{V}{U} \quad (24)$$

When an investor deposits money to or withdraws money from the fund, the number of units changes. If V' is the market value of the portfolio immediately after the cash flow, then the new number of units U' is calculated from the NAV at that time:

$$U' = \frac{V'}{\text{NAV}}$$

The NAV of the unit does not change because of the cash flow—the number of units adjusts to maintain the identity $\text{NAV} = V/U$.

To illustrate this concept further, note that the change in the number of units resulting from a cash flow will be equal to the value of the cash flow divided by the NAV.

$$\Delta U = \frac{\text{CF}}{\text{NAV}}$$

Now consider two periods with an intervening cash flow. Returns calculated from the NAV can be linked over the two periods:

$$R = \frac{\text{NAV}_2}{\text{NAV}_0} - 1 = \left(\frac{\text{NAV}_1}{\text{NAV}_0} \times \frac{\text{NAV}_2}{\text{NAV}_1} \right) - 1 = \left(\frac{V_1/U_1}{V_0/U_0} \times \frac{V_2/U_2}{V_1'/U_1'} \right) - 1$$

In the expression on the right, we have substituted the identity $\text{NAV} = V/U$. Now, because the value V_1 is taken *immediately before the cash flow*, the number of units U_1 is equal to U_0 and cancels in the expression. Similarly, because the value V_1' is

taken *immediately after the cash flow*, the number of units U_2 is equal to U_1' . Therefore, the return calculated from the NAV is equal to the time-weighted return defined above:

$$R = \frac{\text{NAV}_2}{\text{NAV}_0} - 1 = \left(\frac{V_1}{V_0} \times \frac{V_2}{V_1'} \right) - 1$$

If there are no cash flows, the formula for time-weighted return is equivalent to the linking formula (Equation 13). All the concepts of Section 3—compounding, arithmetic and geometric averaging, annualizing, and so on—are applicable. The interpretation of cumulative return, however, must be understood in terms of the growth of a unit value, instead of the growth of the total value of the fund, which is affected by cash flows.

EXAMPLE 28

Unit Value Pricing Method: Calculation

Using the data in Example 27, Question 2, assume that the number of units at the start is 10 million. Calculate the number of units added at each cash flow and the NAV at the start of the month, at each transaction, and at the end of the month.

Solution:

Date (t)	Market Value (millions)	Units before Flow (millions)	NAV	External Cash Flow (millions)	Added Units (millions)
30 Apr	\$73.7	10.0000	$\$73.7/10.000 = 7.3700$		0.0000
12 May	\$69.3	10.0000	$\$69.3/10.000 = 6.9300$	\$15.3	$15.3/6.93 = 2.2078$
17 May	\$87.3	12.2078	$\$87.3/12.2078 = 7.1512$		0.000
26 May	\$89.7	12.2078	$\$89.7/12.2078 = 7.3478$	-\$8.1	$-8.1/7.3478 = -1.1024$
31 May	\$84.7	11.1054	$\$84.7/11.1054 = 7.6269$		0.0000

The cumulative return calculated from NAV is $(7.6269 - 7.3700)/7.3700 = 3.49\%$, which matches the conclusion that we derived in the solution to Question 2 of Example 27.

4.3 Practical Issues

The following sections address several key subtopics in time-weighted rate of return.

4.3.1 When Is Time-Weighted Return Appropriate?

A key difference between the time-weighted return and the internal rate of return is the treatment of external cash flows. Often, external cash flows result from decisions unrelated to the management of the portfolio; deposits and withdrawals by investors may be independent of decisions made by the portfolio manager. To usefully compare returns among alternate portfolios, either the pattern of cash flows must be the same for all the portfolios or the return measurement must be insensitive to cash flows. Consequently, time-weighted return is the preferred measure of return for comparing returns among alternative portfolios.

This point was emphasized in Cohen, Dean, Durand, Fama, Fisher, Shapiro, and Lorie (1968). This study, known as the BAI study, applied the term “time-weighted” to the method of measurement proposed by Dietz (1966). The BAI study described the internal rate of return (IRR) as a return that “measures the performance of the fund rather than the performance of the fund’s manager.”

Time-weighted methodologies are certainly preferred where there is a need to neutralize the impact of external cash flows. Money-weighted methodologies are typically used

- when accurate valuations and, therefore, time-weighted returns are not available (for example, for illiquid asset categories).
- when the manager is responsible for the timings of cash flows.
- when the time value of money is an important issue (for example, in a fund with a fixed termination date, an earlier distribution of capital is likely to improve the internal rate of return).

From the perspective of a client who directs the deposits to and withdrawals from a portfolio, the internal rate of return may be informative because it reflects the decisions made by the client. In some situations (in managing private equity, for example), the portfolio manager does direct the cash flows into the portfolio; in which case, the internal rate of return is an appropriate measure.

4.3.2 Approximate Time-Weighted Return Methods

A drawback of the time-weighted return is that an accurate valuation must be available at the time of all cash flows. Such valuations may be too expensive to obtain or simply unavailable. An approximation of the true time-weighted return must be used when the valuations are not available at the time of cash flows. The basic approximation procedure involves these steps:

- 1 Divide the overall time period into sub-periods corresponding to the availability of valuations. Typically, valuations are done at regular intervals, such as quarterly, monthly, or daily. (Note: For periods beginning on or after 1 January 2001, the GIPS standards require portfolios to be valued at least monthly.)
- 2 Calculate an approximate return for each sub-period.
- 3 Link the sub-period returns.

This procedure was first proposed by Dietz (1966). Variants of the method depend on the approximate return calculated in Step 2.¹⁹

4.3.2.1 Linked IRR Method (BAI Method) In step 2 of the basic approximation procedure, the sub-period return can be calculated using the IRR. In this case, the overall method is called the **linked IRR method**. This method is recommended in the Bank Administration Institute (BAI) study (Cohen et al. 1968) and is thus sometimes called the **BAI method**. Because the linked IRR method combines money-weighted sub-period returns by linking (which is the essence of time weighting), the method may be considered a hybrid method.

The linked IRR method effectively assumes that return is constant over a sub-period. As an approximation to a true time-weighted return, the result is reasonably insensitive to cash flows. The approximation degrades as the magnitudes of the cash flows increase. The BAI study recommended that if a cash flow exceeds 10 percent of the portfolio value, then the portfolio should be revalued to improve the accuracy of

¹⁹ Other approximation procedures that have been proposed are the index substitution method, the regression method, and the analyst’s test method. See Bacon (2008). All three of these methods have largely fallen out of favor over the years.

the performance measurement. (Note: Effective 1 January 2010, the GIPS standards require managers to define what constitutes a large cash flow within the company and revalue the portfolio if the portfolio experiences a large cash flow.)

EXAMPLE 29

Linked IRR Method: Calculation

Calculate the quarterly linked IRR return of a portfolio with the following beginning and ending market values and external cash flows (dollar amounts are in millions):

Date (t)	Market Value (V_t)	External Cash Flow
31 March	\$100.3	
26 April		\$13.8
30 April	\$125.6	
03 May		\$17.8
22 May		-\$25.3
31 May	\$103.5	
18 June		\$15.6
30 June	\$142.7	

Solution:

Calculate IRR returns for each month.

$$\text{April: } 125.6 = 100.3 \times (1 + R) + 13.8 \times (1 + R)^{4/30}$$

$$R = 11.27\% \text{ satisfies the above equation: } 100.3 \times 1.1127 + 13.8 \times (1.1127)^{4/30} = 125.6$$

$$\text{May: } 103.5 = 125.6 \times (1 + R) + 17.8 \times (1 + R)^{28/31} - 25.3 \times (1 + R)^{9/31}$$

$$R = -10.89\% \text{ satisfies the above equation: } 125.6 \times 0.8911 + 17.8 \times (0.8911)^{28/31} - 25.3 \times (0.8911)^{9/31} = 103.5$$

$$\text{June: } 142.7 = 103.5 \times (1 + R) + 15.6 \times (1 + R)^{12/30}$$

$$R = 21.58\% \text{ satisfies the above equation: } 103.5 \times 1.2158 + 15.6 \times (1.2158)^{12/30} = 142.7$$

Linking monthly returns, we find that the linked IRR for the second quarter equals $1.1127 \times 0.8911 \times 1.2158 - 1 = 20.55\%$.

4.3.2.2 Linked Modified Dietz Method Dietz (1966) originally proposed a calculation for Step 2 of the basic approximation procedure that split the cash flows within a sub-period into two equal parts and assigned the pieces to the beginning and the end of the sub-period. Thus, half of the cash flow is added to the beginning value V_0 and half is subtracted from the ending value V_1 . The simple or **original Dietz formula** is

$$R = \frac{V_1 - C/2}{V_0 + C/2} - 1$$

where C is the sum of all cash flows during the sub-period. Equivalently, this formula can be written as

$$R = \frac{V_1 - V_0 - C}{V_0 + C/2}$$

The obvious flaw in the simple Dietz formula is that it makes a general assumption that all cash flows occur at the midpoint of the time period being evaluated. If cash flows are actually concentrated closer to the beginning or end of the time period, the simple Dietz formula can produce distorted results. As a preferred alternative, if the timing of cash flows is available, then the simple Dietz formula can be modified to incorporate that level of detail. The **modified Dietz formula**²⁰ is

$$R = \frac{V_1 - V_0 - \sum_{k=1}^K C_k}{V_0 + \sum_{k=1}^K W_k C_k} \quad (25)$$

where W_k is the day-weighting fraction given by Equation 21. The modified Dietz formula is, in fact, a first-order approximation of the internal rate of return and is, therefore, a money-weighted return calculation for a single sub-period. (The same cannot be said for the simple Dietz formula because it does not accurately weight external cash flows over the performance period.) The fact that the modified Dietz formula is a first-order approximation of the internal rate of return can be seen by solving Equation 25 for V_1 and combining terms, leaving

$$V_1 = V_0 \times (1 + R) + \sum_{k=1}^K C_k \times (1 + W_k R)$$

which can then be compared with Equation 20.

In the modified Dietz formula, the weighted cash flows $\sum_{k=1}^K W_k C_k$ comprise a **beginning adjustment**, from which an **adjusted beginning value** can be calculated:

$$V_0^{adj} = V_0 + \sum_{k=1}^K W_k C_k \quad (26)$$

Similarly, by calculating a corresponding **adjusted ending value**,

$$V_1^{adj} = V_1 - \sum_{k=1}^K (1 - W_k) C_k \quad (27)$$

the modified Dietz formula can be written as

$$R = \frac{V_1^{adj}}{V_0^{adj}} - 1 = \frac{V_1^{adj} - V_0^{adj}}{V_0^{adj}}$$

completely analogous to the holding period return formula (Equation 1). If the modified Dietz formula is used to calculate the sub-period returns, which are then linked, the result is called the **linked modified Dietz method**. The linking process effectively creates a time-weighted return that approximates to a true time-weighted return.

EXAMPLE 30

Linked Modified Dietz Method: Calculation

Using the data from Example 29, calculate the quarterly linked modified Dietz return.

²⁰ Dietz and Kirschman (1983) called this formula the **day-weighted Dietz formula**.

Solution:

Calculate modified Dietz returns for each month:

$$\text{April } \frac{125.6 - 100.3 - 13.8}{100.3 + 13.8 \times \frac{4}{30}} = 11.26\%$$

$$\text{May } \frac{103.5 - 125.6 - 17.8 + 25.3}{125.6 + 17.8 \times \frac{28}{31} - 25.3 \times \frac{9}{31}} = -10.87\%$$

$$\text{June } \frac{142.7 - 103.5 - 15.6}{103.5 + 15.6 \times \frac{12}{30}} = 21.51\%$$

By linking monthly returns for the second quarter, we obtain $1.1126 \times 0.8913 \times 1.2151 - 1 = 20.50\%$. Over monthly periods with steady patterns of incremental cash flows, the IRR rarely differs very much from the modified Dietz return. Hence, the linked modified Dietz and linked IRR methods will tend to generate very similar results.

As mentioned previously, both approximate methods will tend to degrade as the magnitudes of the cash flows increase.

EXAMPLE 31**Approximate Methods Compared with True Time-Weighted Return**

Suppose the data from Example 29 are supplemented with valuations at the times of the cash flows (dollar amounts are in millions):

Date (t)	Market Value before		Market Value (V_t) after	
	Cash Flow	Cash Flow	Cash Flow	Cash Flow
31 March	\$100.30		\$100.30	
26 April	\$114.10	\$13.80	\$127.90	
30 April	\$125.60		\$125.60	
3 May	\$190.50	\$17.80	\$208.30	
22 May	\$260.20	-\$25.30	\$234.90	
31 May	\$103.50		\$103.50	
18 June	\$120.60	\$15.60	\$136.20	
30 June	\$142.70		\$142.70	

Calculate the true time-weighted returns for the months and for the full period. Compare the results with those from the linked IRR and linked modified Dietz methods.

Solution:

$$\text{April: } (114.10/100.30) \times (125.60/127.90) - 1 = 11.71\%$$

$$\text{May: } (190.50/125.60) \times (260.20/208.30) \times (103.50/234.90) - 1 = -16.52\%$$

$$\text{June: } (120.60/103.50) \times (142.70/136.20) - 1 = 22.08\%$$

$$\text{Second quarter: } (1 + 0.1171) \times (1 - 0.1652) \times (1 + 0.2208) - 1 = 13.85\%$$

The true time-weighted return was 13.85 percent, compared with 20.50 percent using linked modified Dietz and 20.55 percent using linked IRR methods. The difference is due to the size of the cash flows, which are consistently large (nearly 10 percent of the portfolio value) and due to the high volatility of the returns.

4.3.3 Consistency of Portfolio Segment Returns with the Overall Portfolio Return

A portfolio can be divided into segments by asset class (stocks, bonds, cash, etc.), by sector, by country, by industry, or in other ways. Each segment can be regarded as a sub-portfolio for which returns can be calculated. The segment weights are the values of the segments as a fraction of the total portfolio value. For example, for a \$200 million portfolio consisting of \$100 million in stocks, \$75 million in bonds, and \$25 million in cash, the segment weights are 50 percent stocks, 37.5 percent bonds, and 12.5 percent cash. Portfolio return is found as the sum of the weighted returns of the segments. Weights and returns of segments may be calculated for a variety of analyses. It is important for weights and returns to be internally consistent: Weights must sum to one, and the weighted returns must sum to the total portfolio return.

With true time-weighted return, consistency is ensured if valuation of the segments coincides with valuation of the total portfolio. For example, suppose a portfolio is divided into two segments, stocks and bonds. Using superscripts to denote the two segments, the total beginning value V_0 is equal to the sum of the beginning value V_0^S of stocks and the beginning value V_0^B of bonds. Similarly, holding period ending values satisfy $V_1 = V_1^S + V_1^B$. The portfolio return can be written as

$$\frac{V_1}{V_0} - 1 = \frac{V_1^S + V_1^B}{V_0} - 1 = \left(\frac{V_0^S}{V_0} \times \frac{V_1^S}{V_0^S} \right) + \left(\frac{V_0^B}{V_0} \times \frac{V_1^B}{V_0^B} \right) - 1 \tag{28}$$

Consequently, the beginning weights $w_S = V_0^S/V_0$ and $w_B = V_0^B/V_0$ are consistent with the segment returns $R_S = (V_1^S/V_0^S) - 1$ and $R_B = (V_1^B/V_0^B) - 1$. Equation 28 can be rewritten as $R = w_S R_S + w_B R_B$.

If there are cash flows without portfolio valuations, then a money-weighted method (modified Dietz or IRR) must be used for the portfolio return. If similar approximations are made for the segment returns, will the beginning weights still be consistent with the returns? Unfortunately, they will not be, in general, as shown by the following example.

EXAMPLE 32

Inconsistency of Beginning Weights with Money-Weighted Returns

Using the data from Example 29 for the month of June, suppose that the portfolio also held a second asset (Asset B) with a beginning value of \$100 million that returned 10 percent for the month. Calculate the IRR and the modified Dietz return for the total portfolio. Further, calculate the weighted sum of the respective returns, using the beginning portfolio weights. (Dollar amounts are in millions.)

Date (t)	Asset A	Asset B	Market Value (V_t)	External Cash Flow
31 May	\$103.5	\$100.0	\$203.5	
18 June				\$15.6
30 June	\$142.7	\$110.0	\$252.7	

Solution:

$$\text{IRR} = 16.04\% \text{ because } 203.5 \times 1.1604 + 15.6 \times (1.1604)^{12/30} = 252.7.$$

$$\text{Modified Dietz return} = \frac{252.7 - 203.5 - 15.6}{203.5 + 15.6 \times \frac{12}{30}} = 16.02\%$$

$$\text{Beginning weights} = \frac{103.5}{203.5} = 50.86\% \text{ and } \frac{100.0}{203.5} = 49.14\%$$

Using the 21.58% IRR calculated in Example 29 for the month of June and the 21.51% modified Dietz return calculated in Example 30,

$$\text{Weighted IRR} = (0.5086 \times 0.2158) + (0.4914 \times 0.1000) = 15.89\% \neq 16.04\%$$

$$\begin{aligned} \text{Weighted modified Dietz return} &= (0.5086 \times 0.2151) + (0.4914 \times 0.1000) \\ &= 15.85\% \neq 16.02\% \end{aligned}$$

The weighted IRR calculation does not equal the portfolio IRR, nor does the weighted modified Dietz return equal the portfolio modified Dietz return, because the beginning weights are not consistent with the returns.

Using the modified Dietz formula, however, weights calculated from the adjusted beginning market values *are* consistent with the returns. To see this, we first note that the internal cash flow C of a segment is equal to

$$C = P - S - I$$

where

P = value of purchases

S = value of sales

I = value of income from assets in the segment

Income appears with a negative sign, as does sales, because both income and sales represent market values that can be used either for purchases within the segment or for transfers to other segments (or withdrawal from the total portfolio). For a given segment, the net value of purchases less sales less income equals the cash inflow to the segment.

Now, a cash flow between two assets creates flows at the segment level of equal value and opposite signs: A sale of one asset creates a purchase of another asset (into the liquidity reserve, perhaps). If all flows are correctly accounted for, then the sum of cash flows over all segments equals the cash flow at the total portfolio level. That is, for a portfolio segmented into stocks and bonds, if ΣC_S and ΣC_B are the sums of all cash flows at a given time into the stock and bond segments, respectively, then the cash flow into the overall portfolio is $C = \Sigma C_S + \Sigma C_B$. Consequently, the adjusted beginning value, Equation 26, and the adjusted ending value, Equation 27, can be segmented, respectively, as $V_0^{adj} = V_0^{adjS} + V_0^{adjB}$ and $V_1^{adj} = V_1^{adjS} + V_1^{adjB}$. The weights $w_S = V_0^{adjS} / V_0^{adj}$ and $w_B = V_0^{adjB} / V_0^{adj}$ are consistent with the returns $R_S = \left(V_1^{adjS} / V_0^{adjS} \right) - 1$ and $R_B = \left(V_1^{adjB} / V_0^{adjB} \right) - 1$. This consistency of weights and returns is a desirable feature of the modified Dietz formula.

Consistency should not be considered a trivial matter in practice. The data available for an analysis may consist of weights and returns that have been previously calculated for various uses. Sometimes weights and returns obtained from different data sources or systems might not be consistent with the return calculated at the total portfolio

level. Simply defining the total portfolio return as the weighted sum of the segment returns may be a useful approach, but it is important to understand that weighted returns might not sum to the true total portfolio return.

EXAMPLE 33

Consistency of Weights and Returns: I

For the data in Example 32, calculate the adjusted beginning weights using the modified Dietz formula, and calculate the weighted sum of the modified Dietz returns. Confirm that the result is equal to the modified Dietz return of the portfolio.

Date (t)	Asset A	Asset B	Market Value (V _t)	External Cash Flow
31 May	\$103.5 million	\$100.0 million	\$203.5 million	
18 June				\$15.6 million
30 June	\$142.7 million	\$110.0 million	\$252.7 million	

Note that the cash flow of \$15.6 million was added entirely to Asset A.

Solution:

The adjusted beginning value of Asset A is $103.5 + 15.6 \times 12/30 = 109.74$. The adjusted beginning value of Asset B is 100.00 (there are no cash flows into Asset B). The adjusted beginning value of the portfolio is $203.5 + 15.6 \times 12/30 = 209.74$. The weighted modified Dietz return is

$$\left(\frac{109.74}{209.74} \times 0.2151 \right) + \left(\frac{100.00}{209.74} \times 0.1000 \right) = 16.02\%$$

Weights and returns calculated from adjusted beginning and ending values using the modified Dietz formula are consistent.

EXAMPLE 34

Consistency of Weights and Returns: II

Building on the data in Example 33, on 10 July the manager sells \$40 million out of Asset B, which was used to purchase an additional \$40 million of Asset A. On 31 July, the position in Asset A is valued at \$187.2 million and the position in Asset B is valued at \$72.4 million. For the month of July, calculate the adjusted beginning weights using the modified Dietz formula, and calculate the weighted sum of the modified Dietz returns. Confirm that the result is equal to the modified Dietz return of the portfolio.

Date (t)	Asset A	Asset B	Market Value (V _t)	Cash Flow from Asset B to Asset A
30 June	\$142.7 million	\$110.0 million	\$252.7 million	
10 July				\$40.0 million
31 July	\$187.2 million	\$72.4 million	\$259.6 million	

Solution:

- The adjusted beginning value of Asset A is $142.7 + 40.0(21/31) = 169.80$.
The modified Dietz return for Asset A is $(187.2 - 142.7 - 40.0)/169.80 = 0.0265 = 2.65\%$.
- The adjusted beginning value of Asset B is $110.0 - 40.0(21/31) = 82.90$.
The modified Dietz return for Asset B is $(72.4 - 110.0 + 40.0)/82.90 = 0.0289 = 2.89\%$.

The adjusted beginning value of the portfolio is $169.8 + 82.9 = 252.7$. Note that this is the same as the beginning portfolio value without adjusting for cash flows ($142.7 + 110.0 = 252.7$). Because the cash flow between Assets A and B was internal, no adjustment at the portfolio level is necessary.

The weighted modified Dietz return is

$$\left(\frac{169.80}{252.70} \times 0.0265 \right) + \left(\frac{82.90}{252.70} \times 0.0289 \right) = 0.0273 = 2.73\%$$

Now, calculate the modified Dietz return of the total portfolio:

$$R = \frac{V_1 - V_0 - \sum_{k=1}^K C_k}{V_0 + \sum_{k=1}^K W_k C_k}$$

In this case, because there are no external cash flows (the cash flow was a transfer between sub-portfolios), the formula can be simplified as

$$R = \frac{V_1 - V_0}{V_0} = \frac{259.6 - 252.7}{252.7} = 0.0273 = 2.73\%$$

Again, weights and returns calculated from adjusted beginning and ending values using the modified Dietz formula are consistent, even after capital has been reallocated across segments.

4.4 Composite Returns

For both internal analysis and external presentation to prospective clients, managers typically group together portfolio returns to create a representative average (or **composite**) return. The GIPS standards define a composite as an aggregation of one or more portfolios managed according to a similar investment mandate, objective, or strategy and require that composite returns be calculated by asset weighting the individual portfolio returns using beginning-of-period market values or a method that reflects both beginning-of-period values and external cash flows.

As with portfolio returns, the more frequent the valuation points used in calculating composite returns, the more accurate the performance results will be. This is complicated by the fact that composites may include dozens (perhaps even hundreds or thousands) of accounts, all of which would need to be valued at the same point in time in order to calculate a true time-weighted rate of return. To reduce the complexity of composite performance calculations, various portfolio weighting methods have been developed. Acceptable methods include using weights calculated with beginning-of-period market values, using weights calculated from the denominator of the modified Dietz formula, or calculating the composite as a single portfolio consisting of all the assets of the constituent portfolios (i.e., the aggregate method).

The beginning market value weights approach is generally the simplest approach because it requires the fewest calculations. For example, consider a composite consisting of two portfolios: Portfolio A, with a beginning-of-month market value of

\$25 million, and Portfolio B, with a beginning market value of \$75 million. Portfolio A had a return for the month of 10 percent, and Portfolio B had a return for the month of 8 percent. Using the beginning-of-period market value approach, the composite return would be calculated as follows:

$$R = \frac{25}{100} \times 10.00\% + \frac{75}{100} \times 8.00\% = 8.50\%$$

The drawback, however, is that the weights of the individual portfolios do not take into account external cash flows throughout the period. When relying strictly on beginning-of-period market value weights, portfolios that experience positive cash flows during the period will be underweighted in the composite return calculation in relation to the assets that actually contributed to the portfolio return; at the same time, accounts that experience withdrawals will be overweighted. This issue is alleviated by applying weights calculated using the modified Dietz method because the weightings include both the beginning-of-period market values and weighted cash flows—although the calculations can get a bit more cumbersome.

Returning to our previous example, assume Portfolio A had experienced a \$10 million contribution on the 12th day of a 31-day month. The composite return using modified Dietz weightings would be

$$\text{Weighted cash flow} = 10 \times \frac{19}{31} = 6.13$$

$$R = \frac{31.13}{106.13} \times 10.00\% + \frac{75}{106.13} \times 8.00\% = 8.59\%$$

The third approach, the aggregate method, compiles all the holdings of individual portfolios into one “super portfolio.” Although mathematically accurate, this approach can be difficult to perform manually, particularly when working with a composite containing a large number of portfolios.

Continuing our example, assuming the end-of-period portfolio values are \$38.12 million and \$81 million, respectively, the composite would be calculated as

$$R = \frac{119.12 - 100 - 10}{100 + 10 \times \frac{19}{31}} = 8.59\%$$

Note that the aggregate method and the method using modified Dietz weightings produce the same result. This will not always be the case, as illustrated in the example below.

EXAMPLE 35

Calculating Composite Returns

Calculate the January composite return for the following constituent portfolios (in millions) using beginning market values, beginning market values plus external cash flows, and the aggregate method.

Portfolio	Market Value 31 Dec	Cash Flow 10 Jan	Cash Flow 18 Jan	Cash Flow 25 Jan	Market Value 31 Jan	Return
A	\$10.2				\$11.7	14.71%
B	\$33.6				\$39.5	17.56%
C	\$16.7	\$0.9			\$20.1	14.44%
D	\$25.1		-\$1.2		\$27.6	15.04%

(continued)

Portfolio	Market Value 31 Dec	Cash Flow 10 Jan	Cash Flow 18 Jan	Cash Flow 25 Jan	Market Value 31 Jan	Return
E	\$151.3			\$25.0*	\$199.7	14.17%
Total	\$236.9				\$298.6	

* Note that the cash flow in Portfolio E on 25 January is defined as a large cash flow for both Portfolio E and in the context of the entire composite.
 Value of Portfolio E on 25 Jan = \$160.2 million
 Value of entire composite on 25 Jan = \$245.7 million

Solution:

For Portfolio E, the true time-weighted return for the month is calculated as follows:

$$R = \frac{160.2}{151.3} \times \frac{199.7}{160.2 + 25.0} - 1 = (1.0588)(1.0783) - 1 = 14.17\%$$

Beginning market value

$$\frac{10.2}{236.9} \times 14.71\% + \frac{33.6}{236.9} \times 17.56\% + \frac{16.70}{236.9} \times 14.44\%$$

$$+ \frac{25.1}{236.9} \times 15.04\% + \frac{151.3}{236.9} \times 14.17\% = 14.79\%$$

Beginning market value plus external cash flow

$$\text{Portfolio A} \quad 10.2 + 0 = 10.2 \quad \frac{10.2}{241.85} = 4.22\%$$

$$\text{Portfolio B} \quad 33.6 + 0 = 33.6 \quad \frac{33.6}{241.85} = 13.89\%$$

$$\text{Portfolio C} \quad 16.7 + 0.9 \times \frac{21}{31} = 17.31 \quad \frac{17.31}{241.85} = 7.16\%$$

$$\text{Portfolio D} \quad 25.1 + -1.2 \times \frac{13}{31} = 24.6 \quad \frac{24.6}{241.85} = 10.17\%$$

$$\text{Portfolio E} \quad 151.3 + 25.0 \times \frac{6}{31} = 156.14 \quad \frac{156.14}{241.85} = 64.56\%$$

$$\text{Total} \quad = 241.85 \quad = 100\%$$

$$(4.22\% \times 14.71\%) + (13.89\% \times 17.56\%) + (7.16\% \times 14.44\%)$$

$$+ (10.17\% \times 15.04\%) + (64.56\% \times 14.17\%) = 14.77\%$$

Aggregate method:

First sub-period return using modified Dietz

$$\frac{245.7 - 0.9 + 1.2 - 236.9}{236.9 + 0.9 \times \frac{15}{25} - 1.2 \times \frac{7}{25}} = 3.84\%$$

Second sub-period return

$$\frac{298.6}{245.7 + 25.0} - 1 = 10.31\%$$

Aggregate method total return: $(1 + 3.84\%) \times (1 + 10.31\%) - 1 = 14.55\%$

Note: The aggregate method produces a result different from the result of the beginning market value plus external cash flow method. This is because the market value plus external cash flow weighting does not take into account the revaluation of Portfolio E.

CONCLUSIONS AND SUMMARY

5

The calculation of the rate of return is complex, with many pitfalls for the unwary. There are many different acceptable calculation methodologies available to analysts resulting from the different treatment of external cash flows. Clearly there is an opportunity to manipulate returns by simply choosing the highest return or self-selecting methodologies in each period. This unethical approach must be avoided; managers should establish an internal policy for performance measurement and apply this policy consistently. Among the points made in this reading are the following:

- A holding period rate of return is the change in value created by holding the portfolio over a specified period of time. The portfolio values at the beginning and at the end of the period are essential data for calculating a holding period return.
- An internal cash flow is a transfer of value from one asset to another within a portfolio, whereas an external cash flow is a transfer of value into or out of a portfolio. Investment income (e.g., dividends or interest payments) may create an internal cash flow if the income is reinvested in the portfolio, or it may create an external cash flow if the income leaves the portfolio.
- Transactions such as purchases, sales, and income received typically create cash flows to or from a liquidity reserve, or cash asset. A portfolio rate of return can be calculated from the total value of the portfolio, which includes the liquidity reserve. The total value must also include accrued dividends and interest belonging to the portfolio.
- A portfolio of assets in multiple currencies must be converted to a base currency using currency exchange rates.
- Market values, rather than accounting concepts of book value, are preferred for calculating rates of return to accurately depict economic value.
- Various types of return can be calculated: nominal versus real, pre-tax versus post-tax, gross-of-fees versus net-of-fees, and leveraged versus cash basis returns.
- Returns over multiple holding periods can be chain linked, or compounded, to produce a cumulative return over a longer period.
- The periodicity of the return is not a characteristic of the investment; it is a parameter chosen by the performance measurer. For analytical purposes, discrete compounding or continuous compounding constitute alternative models of growth. The models are equivalent if portfolio values match at the given valuation time points.
- Two types of averages, arithmetic and geometric, are useful. Arithmetic averages can be used for statistical modeling and forecasting, having well-known properties. Geometric averages are used for accurately summarizing historical returns. Annualizing should be done only if the returns span a period longer than a year.

- An excess return is the difference between a portfolio return and a benchmark return. Excess returns can be calculated as either arithmetic differences or geometric differences.
- External cash flows complicate the calculation of returns because of their potential to distort performance results.
- Time-weighted rates of return are insensitive to external cash flows. The calculation requires valuations at the times of each cash flow. The time-weighted return can be interpreted as the growth of a unit value if the portfolio was treated as a mutual fund.
- Money-weighted rates of return are affected by external cash flows and may be appropriate if the cash flows are under the control of the portfolio manager.
- Because external cash flows are usually not under the control of the portfolio manager, time-weighted returns are preferred for comparing the performance of different funds.
- True time-weighted return can be approximated by valuing the portfolio periodically, calculating a money-weighted return for each period between valuations, and linking the results.
- If the sub-period returns are calculated using the internal rate of return (IRR), the method is called the linked IRR method. If the sub-period returns are calculated using the modified Dietz formula, the method is called the linked modified Dietz method.
- A portfolio can be divided into segments. If there are no cash flows during a period, returns calculated from the segments and weighted by the beginning weights will equal the return of the total portfolio. This condition is called consistency of weights and returns.
- If there are cash flows during a period, segment returns calculated with a money-weighted method will, in general, not be consistent with segment returns calculated with beginning weights. However, adjusted beginning weights and returns calculated with the modified Dietz formula are consistent and remain so as capital is allocated across portfolio segments.
- The GIPS standards prescribe methods of weighting multiple portfolios of a given portfolio management firm into a composite return.

APPENDIX

OPTIONAL
SEGMENT

This appendix explains and concisely illustrates basic terminology and notation used in the “Rate-of-Return Measurement” reading. It should be helpful for candidates who do not have a background in investments.

1 Mathematical Notation

In this section, we discuss the use of summation and product operators. These operators are symbols used to represent mathematical operations—such as addition and multiplication—in a concise manner.

1.1 The Summation Operator

The summation operator, represented by the Greek letter sigma, Σ , can be used to represent an equation involving the adding (summation) of variables. Consider the following equation:

$$y = x_1 + x_2 + x_3 + x_4 + x_5$$

This equation can be represented using the summation operator as follows:

$$y = \sum_{t=1}^5 x_t$$

The notation below and above sigma indicates how the addition can be performed as a set of steps. First, the index t is set equal to 1, resulting in $y = x_1$; t is then incremented to 2, resulting in $y = x_1 + x_2$; then t is incremented through 3, 4, and 5 (the limit indicated on top of sigma), resulting finally in $y = x_1 + x_2 + x_3 + x_4 + x_5$.

A memory aid is “the s in sigma stands for *sum*.”

EXAMPLE A1

Calculating the Arithmetic Mean

The arithmetic mean of a set of items is equal to the sum of the items divided by the number of items summed. Given the following return data,

Year	Return
2006	10.6%
2007	2.5%
2008	0.5%
2009	24.5%
2010	14.2%

- 1 Give the expression for the arithmetic mean return using summation notation. Let R_t stand for the return in year t , and let T be the number of returns to be summed.
- 2 Calculate the arithmetic mean (or average) of the series.

Solution to 1:

Returns for five years are involved, so the formula to calculate the arithmetic mean return is

$$\text{Arithmetic mean return} = \frac{\sum_{t=1}^5 R_t}{T}$$

Solution to 2:

$$\text{Arithmetic mean return} = \frac{10.6\% + 2.5\% + 0.5\% + 24.5\% + 14.2\%}{5} = 10.46\%$$

1.2 The Product Operator

The product operator, represented by the Greek letter pi, Π , is used to represent an equation involving the multiplication of a number of variables. For example, consider the following equation:

$$z = w_1 \times w_2 \times w_3 \times w_4 \times w_5$$

This equation can be represented in a concise manner using the product operator as follows:

$$z = \prod_{t=1}^5 w_t$$

A memory aid is “the *p* in pi stands for *product*.” (*Product* is the result of a multiplication.)

EXAMPLE A2

Calculating the Geometric Mean

The formula to calculate the geometric mean return is

$$\text{Geometric mean return} = \left[\prod_{t=1}^5 (1 + R_t) \right]^{1/T} - 1$$

where R_t is the return for each year and T is the number of periods used in the calculation. Calculate the geometric mean return of the following series:

Year	Return
2006	20.6%
2007	22.5%
2008	7.5%
2009	4.5%
2010	10.2%

Solution:

Representing the annual returns in decimal form (e.g., 20.6% is written 0.206 in decimal form), the geometric mean return is

Geometric mean return

$$= (1.206 \times 1.225 \times 1.075 \times 1.045 \times 1.102)^{1/5} - 1$$

$$= 0.1283 = 12.83\%$$

2 Debt and Equity Securities: Overview

The debt market is a market for borrowing and consists of the bond market and the money market. Instruments that trade in the bond market have maturities that exceed one year, whereas money market instruments have maturities of one year or less. Governments and businesses are leading issuers of bonds. The issuer of a bond promises certain payments to bondholders specified in the bond's contract. Those promised payments are legal obligations of the bond issuer.

Equity securities can be categorized as common stock or preferred stock. Businesses are leading issuers of equity securities. In either case, any payments made to equity are not legal obligations of the equity issuer. Furthermore, payments to bond holders have priority over payments to equity holders. Common stock represents an ownership right in a company.

Preferred stock has characteristics of both debt and common stock. Like a debt instrument, preferred shareholders expect to receive a fixed payment and receive no voting rights. However, like common stock, the payment of dividends in a particular year is discretionary, although preferred dividends must be paid before any common share dividends are paid. If preferred dividend payments are skipped, they typically accumulate and must be paid before any common share dividends are paid out.

3 Debt Securities

In this section, we discuss basic concepts and calculations related to various types of debt securities.

3.1 Bonds and Notes

A bond represents an instrument that is issued as part of a borrowing agreement. The bond has a par value that represents the amount that must be repaid by the issuer to the lender when the bond matures. The par value is also referred to as the face value, principal value, redemption value, or maturity value. It is common practice to quote bond prices as a percentage of par value. Examples of bond quotes are provided in a separate section below.

Bonds generally trade in electronic (i.e., online) trading systems ("platforms") and in dealer markets (in which dealers stand ready to buy from investors who want to sell and then sell to investors who want to buy).

In most cases the issuer will pay the lender a periodic fixed interest payment known as the coupon. The coupon rate is quoted as an annual percentage of par value. The annual coupon payment is calculated as

$$\text{Annual coupon payment} = \text{Annual coupon rate} \times \text{Par value}$$

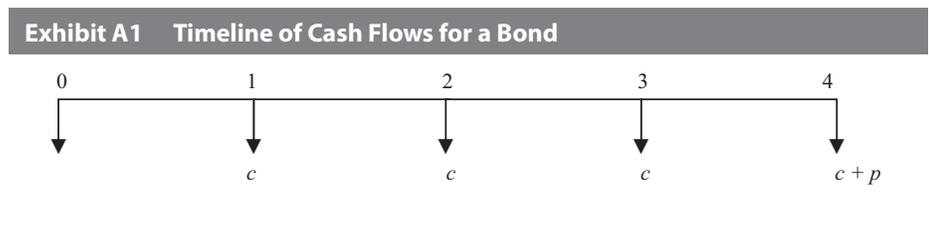
For example, if a bond has a par value of \$1,000 and pays an annual coupon rate of 4.75 percent, then the annual coupon is $0.0475 \times \$1,000 = \47.50 .

If the bond pays interest every six months, as is the case in the United States, then the semi-annual coupon is calculated as follows:

$$\text{Semi-annual coupon} = \frac{\text{Annual coupon rate} \times \text{Par value}}{2}$$

The term to maturity of a bond is the number of years remaining until the principal value of the bond (or par value) is repaid to the lender. On that date, the debt ceases to exist. It is common practice to refer to the term to maturity of a bond as the maturity or term. Bonds with maturities of 1 to 5 years are sometimes referred to as short term, whereas bonds with maturities between 5 and 12 years are considered intermediate term. Bonds with maturities that exceed 12 years are considered long term. In the US Treasury market, securities with maturities of 1 to 10 years are called notes, whereas those with maturities of more than 10 years are called bonds.

Exhibit A1 below provides a summary of cash flows associated with a bond that pays an annual coupon of \$*c*, has a par value of \$*p*, and matures in four years.



EXAMPLE A3

Treasury Bond Quotes

Exhibit A2 provides a quote for a US Treasury bond from the *Wall Street Journal* (1 June 2011).

Exhibit A2 Treasury Bond Quote

Maturity	Coupon	Bid	Asked	Chg	Asked yield
2041 May 15	4.375	102:22	102:23	+14	4.2152

This quote indicates that the bond matures 15 May 2041 and pays an annual coupon of 4.375 percent. In the United States, most bonds typically have a face value of \$1,000 and pay semi-annual interest. In this case the purchaser of the bond will receive \$21.875 every six months: $(0.04375/2) \times \$1,000 = \21.875 .

In the Treasury market, prices are quoted in 32nds.²¹ The bid price of 102:22 is interpreted as $102\frac{22}{32}$ percent of \$1,000. This means that an investor wishing to sell this security would receive \$1,026.875. In a similar manner, an investor wishing to purchase this bond would have to pay \$1,027.1875; that is, the purchase price is $102\frac{23}{32}$ percent of \$1,000. The next column indicates that the price of the bond rose by \$4.375 or $\frac{14}{32}$ percent of \$1,000. The last column, titled *Asked yield*, indicates that an investor purchasing this bond can expect to earn an annualized yield (or yield to maturity) of 4.2152 percent. A later section provides more detail on the interpretation and calculation of the yield to maturity.

²¹ Quotation in 64ths and even 256ths is also possible.

EXAMPLE A4

Corporate Bond Quotes

Exhibit A3 provides a quote for a corporate bond obtained from the *Wall Street Journal* (1 June 2011).

Exhibit A3 Corporate Bond Quote									
Issuer Name	Coupon	Maturity	Rating		High	Low	Last	Change	Yield
			Moody's/ S&P/ Fitch						
WELLS FARGO & CO	5.625%	Dec 2017	A1/AA-	AA-	113.094	111.500	112.793	0.293	3.421

Assuming a par value of \$1,000, the bond pays interest of \$28.125 every six months, calculated as $(0.05625/2) \times \$1,000$. The bond matures in December 2017. Corporate bonds receive a credit rating from three credit-rating agencies: Moody's, Standard & Poor's, and Fitch. These ratings are presented in the next column.

The next two columns indicate that the bond traded in a range of \$1,130.94 to \$1,115.00 during the day.²² The bond closed at a price of \$1,127.93. The change in price from open to the time of the quote is 0.293 as a percentage of par, or \$2.93. The last column provides the yield to maturity of the bond. That is, the investor can expect to earn an annualized rate of approximately 3.421 percent if this bond is held to maturity.

3.1.1 Calculating the Value of a Bond on a Coupon Date

The value of a bond is the present value of future expected cash flows, to maturity, discounted by the bond's periodic yield to maturity i . Denoting the periodic coupon payment as C , the maturity as N , and the par value as P , the equation to calculate the value of a bond is

$$\text{Value} = \frac{C}{(1+i)^1} + \frac{C}{(1+i)^2} + \frac{C}{(1+i)^3} + \dots + \frac{C}{(1+i)^N} + \frac{P}{(1+i)^N}$$

Or more compactly, using the summation notation Σ , the valuation equation is

$$\text{Value} = \sum_{n=1}^N \frac{C}{(1+i)^n} + \frac{P}{(1+i)^N}$$

A bond that sells at face value is referred to as a par value bond, and a bond that trades at a price above face value is referred to as a premium bond. A bond that trades at a price below face value is referred to as a discount bond.

²² Unlike the previous Treasury bond example, the quotes in this example are in cents (not in 32nds).

EXAMPLE A5**Bond Valuation on Coupon Date**

By way of illustration, consider the following example. A bond has an annual coupon of 8 percent paid semi-annually. The bond matures in two years and has an annual yield to maturity of 10 percent. The par value is \$1,000. The value of the bond is the present value of future expected payments:

$$\begin{aligned} \text{Value} &= \frac{40}{(1 + 0.05)^1} + \frac{40}{(1 + 0.05)^2} + \frac{40}{(1 + 0.05)^3} + \frac{40}{(1 + 0.05)^4} + \frac{1,000}{(1 + 0.05)^4} \\ &= \$964.54 \end{aligned}$$

The semi-annual coupon payment C is $(0.08 \times \$1,000)/2 = \40 . The periodic yield to maturity i is $(0.10/2) = 0.05$. The maturity N is four six-month periods.

3.1.2 When a Bond Is Sold between Coupon Payment Dates

When a bond is sold between coupon dates, the buyer of the bond will receive the full coupon on the next coupon payment. Because the buyer has not owned the bond through the full coupon period, the buyer must compensate the seller in an amount known as accrued interest. Accrued interest is proportional to the fraction of days between coupon payment dates that the bond was held by the seller. For example, if that fraction is $\frac{1}{4}$ and the coupon is \$60, the amount of accrued interest is $(1/4)(\$60) = \15 . However, quoted bond prices do not generally include accrued interest. The quoted price is called the clean price. The clean price plus accrued interest is known as the full price or dirty price and is the amount of money actually received for the bond by the seller (apart from sales-related costs such as commissions). Thus, if the clean price was \$920 and accrued interest was \$15, the dirty price would be \$935.

EXAMPLE A6**Bond Valuation between Coupon Dates**

Suppose the quoted price of a bond is \$900 and the bond pays an annual coupon of \$50. The bond is sold halfway through coupon payment dates. Calculate the dirty price of the bond.

Solution:

The dirty price of the bond is $\$900 + (1/2)(\$50) = \$925$.

3.1.3 Yield-to-Maturity Calculation

The yield to maturity of a bond is the approximate average annual rate of return that an investor can expect to earn on a bond purchased today and held to maturity. It is also the discount rate that equates the present value of a bond's future cash flows to its price. Recall that we previously stated that the value of a bond is calculated as follows:

$$\text{Value} = \frac{C}{(1 + i)^1} + \frac{C}{(1 + i)^2} + \frac{C}{(1 + i)^3} + \dots + \frac{C}{(1 + i)^N} + \frac{P}{(1 + i)^N}$$

The yield to maturity in this equation is i . Given the price of the bond, the coupon payment, the face value, and the periods to maturity, the equation can be solved to obtain the yield to maturity. The yield to maturity consists of returns attributed to coupon payments as well as the capital gains or losses.

Yield to maturity = Current yield + Capital gains/losses yield

where Current yield = Coupon payment/Price of bond.

EXAMPLE A7

Yield to Maturity for a Discount Bond

A bond sells for \$975, carries an annual coupon rate of 5 percent paid semi-annually, matures in three years, and has a face value of \$1,000. The yield to maturity for this bond can be solved using the bond valuation equation provided above.

$$975 = \frac{25}{(1+i)^1} + \frac{25}{(1+i)^2} + \frac{25}{(1+i)^3} + \frac{25}{(1+i)^4} + \frac{25}{(1+i)^5} + \frac{1,025}{(1+i)^6}$$

Solving for i using a financial calculator or Microsoft Excel, we find that $i = 2.96\%$. The bond's annualized yield to maturity is $2.96\% \times 2 = 5.92\%$.

The yield to maturity can be decomposed into the current yield and capital gains yield as follows:

$$5.92\% = 5.13\% + 0.79\%$$

Current yield = Coupon payment/Price of bond = $50/975 = 5.13\%$. The capital gains yield is 0.79 percent and reflects the fact that the bond has been purchased at a discount to face value (\$975) and will pay \$1,000 at maturity.

3.2 Money Market Instruments

Money market instruments are highly marketable debt instruments with maturities not exceeding one year. Examples of money market instruments include US Treasury bills, commercial paper, bankers' acceptances, and repos or repurchase agreements. Most money market instruments carry a minimum face value of \$100,000. In the sections that follow, we provide a more detailed discussion of two well-known money market instruments: US Treasury bills and commercial paper.

3.2.1 Treasury Bills

The US government borrows money for short periods by issuing T-bills with initial maturities of 4, 13, 26, or 52 weeks. T-bills pay the full face value at maturity but do not make coupon payments. Investors purchase these instruments at a discount to face value, which is typically \$10,000. The difference between the purchase price and maturity value represents the investor's return from implicit interest.

Unlike Treasury bonds and corporate bonds, T-bills are quoted using a bank discount method. A T-bill quote from the *Wall Street Journal* (2 June 2011) is presented below.

Exhibit A4 Treasury Bill Quote

Maturity	Bid	Asked	Chg	Asked yield
2012 Mar 08	0.128	0.120	-0.003	0.121

The first column indicates that the T-bill matures 8 March 2012, which is 280 days from the date of the quote, 2 June 2011. The bid column indicates that the bid price or sale price is approximately 0.128 percent below the face value of \$10,000, whereas the asked column indicates that the asked price or purchase price is approximately 0.12 percent below the face value of \$10,000. The last column, the asked yield, is the annualized rate of return to the investor purchasing the bond today. It is also referred to as the bond equivalent yield. The asked price of the T-bill can be calculated using the following equation:

$$\text{Discount yield} = \frac{\text{Face value} - \text{Asked price}}{\text{Face value}} \times \frac{360}{\text{Days to maturity}}$$

We substitute the relevant terms into the equation:

$$0.0012 = \frac{\$10,000 - \text{Asked price}}{\$10,000} \times \frac{360}{280}$$

Solving for the asked price, we obtain

$$\text{Asked price} = \$10,000 - \left[\$10,000 \times 0.00120 \times \frac{280}{360} \right] = \$9,990.67$$

The asked yield or bond equivalent yield is then calculated using a 365-day year as²³

$$\begin{aligned} \text{Asked yield} &= \frac{\text{Face value} - \text{Asked price}}{\text{Asked price}} \times \frac{365}{\text{Days to maturity}} \\ &= \frac{\$10,000 - \$9990.67}{\$9990.67} \times \frac{365}{280} = 0.00121 = 0.121\% \end{aligned}$$

(or 0.122 percent based on using the rounded numbers given in the asked yield calculation).

3.2.2 Commercial Paper

Large, well-known companies issue short-term unsecured debt instruments called commercial paper. Maturities can be as long as 270 days, but they generally average about 30 days. These instruments typically have a face value of \$100,000 and are a low-cost alternative to bank loans. Commercial paper issuance falls into one of four categories:

- AA non-financial
- A2/P2 non-financial
- AA financial
- AA asset-backed

The letters indicate perceived creditworthiness. These instruments trade in very liquid secondary markets and carry annualized yields in the range of 0.09 percent to 0.36 percent depending on maturity.

²³ By money market convention, a 360-day year is frequently used to price contracts. However, the analyst can use the actual number of days in a year to most accurately determine the return or yield.

4 Equity Securities

In this section, we cover fundamental concepts regarding equity securities. We review a basic stock quote, the dividend payment chronology, valuation, and the basics of stock market index construction.

4.1 Common Shares

Common shares represent an ownership stake in a company. Common shareholders are entitled to vote at the company's annual meeting. At the discretion of the company, the shareholder may also receive a dividend payment. Common shareholders have a residual claim on the company's assets and income. That is, should the company be liquidated, common shareholders are the last in line to receive payments after all other claimants have been paid.

Common shareholders enjoy the benefit of limited liability. Should the company fail, the most that a common shareholder can lose is the original investment; that is, they are not liable for any liabilities of the company.

In the United States, stocks trade on exchanges, such as the NYSE and the NASDAQ, as well as on electronic communication networks (ECNs). We provide a quote below for IBM from the NYSE.

EXAMPLE A8

Stock Quote

Exhibit A5 below provides a stock quote from the *Wall Street Journal* (3 June 2011).

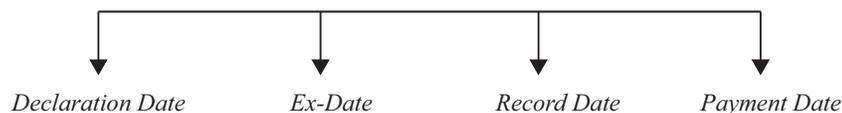
Exhibit A5 Stock Quote							
	Open	High	Low	Close	Net Chg	% Chg	
International Business Machines	166.44	167.10	165.71	166.09	-0.47	-0.28	
	Vol	52 Week High	52 Week Low	Div	Yield	PE	YTD % Chg
International Business Machines	3,847,659	173.54	120.61	3.00	1.81	13.95	13.17

The stock opened trading at \$166.44 and traded between \$167.10 and \$165.71 during the day, with the last trade taking place at \$166.09. The stock pays an annual dividend of \$3 per share. The dividend yield is 1.81 percent and is calculated by dividing the dividend per share by the price. The quote also indicates that the price-to-earnings (P/E) multiple is 13.95; that is, it trades at 13.95 times 12-month trailing earnings per share.

The main sources of return for a stock investment are price changes (capital gains/losses) and dividend income. The frequency of dividend payments during a year varies by country. Most companies in the United States and Canada pay dividends once every three months. European and Japanese companies typically pay dividends every six months. Annual dividend payments are common for Chinese companies.²⁴

The chronology of dividend payments includes several notable dates: the declaration date, the ex-dividend date, the holder-of-record date, and the payment date. The declaration date is the day that the company announces a specific dividend payment. The ex-dividend date, or ex-date, is the first date that the share trades without the right to receive the next dividend. Investors who own the share before the ex-date receive the dividend; however, investors who purchase the share on or after the ex-dividend date will not receive the dividend. The holder-of-record date is two business days after the ex-dividend date and is the date that a shareholder is deemed by the company to have ownership of the stock. The payment date is the date the company makes the dividend payment. Exhibit A6 below provides a timeline of the dividend payment chronology.

Exhibit A6 Dividend Payment Timeline



A widely used equity valuation model is Gordon's constant growth model, sometimes referred to as the dividend discount model (DDM). It is a type of a discounted cash flow (DCF) model. Gordon's model maintains that the intrinsic value of a stock is the present value of all future dividend payments, assuming that they grow at a constant rate of g in perpetuity. Under these assumptions, the value, V_0 , is calculated as

$$V_0 = \frac{D_1}{k - g}$$

where D_1 is the dividend payment expected next year and k is the required rate of return on the stock.

EXAMPLE A9

Calculating the Intrinsic Value of a Stock

A stock is expected to pay a dividend of \$1.50 per share next year. Dividends are expected to grow in perpetuity at a constant rate of 2.5 percent per year. The required rate of return on this stock is 7.25 percent. The intrinsic value according to Gordon's constant growth model is

$$V_0 = \frac{1.50}{0.0725 - 0.025} = \$31.5$$

²⁴ George H. Troughton and Gregory Noronha, "Dividend and Share Repurchases: Basics" (Charlottesville, VA: CFA Institute, 2010).

4.1.1 Stock Market Indexes

A stock market index can be used to gauge the performance of overall equity markets or an equity market segment. In the United States, well-known stock market indicators include the Dow Jones Industrial Average (DJIA) and the S&P 500. The DJIA is a price weighted average of 30 large “blue chip” companies. This average is calculated by adding up the prices of the 30 shares and dividing by a divisor D that has been adjusted for stock splits and changes in the constituents of the 30 companies. The DJIA measures the return on a portfolio that holds one share of each of the 30 constituent stocks. The weight of each stock in the index is a function of the price of the stock.

The S&P 500 composite is a market value weighted index, where the market value of a stock is the stock price times the number of shares outstanding. This index measures the return of a portfolio of the 500 companies invested in proportion to each company’s market value.

Stock indexes can provide a measure of price return or total return. A price return index reflects only changes in the prices of the stocks in the index, whereas a total return index measures changes in the prices of the stocks in the index as well as the reinvestment of dividend income.

Generally, the value of a price return index is calculated as follows:

$$\text{Price return index} = V^x = \frac{\sum_{i=1}^N n_i P_i}{D}$$

where the superscript x indicates that it is the value of the index without dividends reinvested.

The price return of the index, R , can be calculated by determining the percentage change in the index—for example, from Time 0 to Time 1:

$$R = \frac{V_1^x - V_0^x}{V_0^x}$$

where V_1^x is the value of the price return index at Time 1 and V_0^x is the value of the price return index at Time 0.

The return for a total return index is the price change for the index plus the return from dividend income generated by the component stocks. The return on a total return index is calculated as follows:

$$\text{Return on total return index} = \frac{V_1 - V_0}{V_0}$$

where V_1 and V_0 represent the values of the total return index (i.e., with dividends reinvested) at Time 1 and Time 0.

4.2 Preference Shares

Preference shares or preferred stock are equity instruments, but they share features of debt and common stock. Like debt instruments, preferred shareholders usually receive fixed payments. However, these dividends are paid at the discretion of the company, much like common shares. Dividends on preferred shares are generally cumulative; that is, if the company skips a dividend payment, it accumulates missed dividends and accumulated dividends must be paid to preference shareholders before the payment of any common share dividends.

5 Foreign Exchange Quotation

Domestic investors investing in foreign markets must account for the impact of currency exchange rates. An exchange rate is the number of units of one currency required to purchase one unit of another currency. An exchange rate can be quoted in two ways. For the US dollar and the euro, the two ways are

- €0.68493 per \$1, which may be written €0.68493/\$1 (or \$:€ = 0.68493).²⁵ This means that €0.68493 buys \$1 or, equivalently, \$1 buys €0.68493.
- \$1.46 per €1, which may be written \$1.46/€1 (or €:\$ = 1.46). This means that \$1.46 buys €1 or, equivalently, €1 buys \$1.46.

The two quotations are reciprocals of each other; that is, $1/0.68493 = 1.46$.

EXAMPLE A10

Currency Appreciation and Depreciation

Suppose a US investor observes that the exchange rate between the US dollar and the UK pound is \$1.64/£. If the exchange rate one week later is \$1.75/£, it implies that the dollar has declined (or depreciated) against the pound because it takes more dollars to purchase one pound. Equivalently, the pound has strengthened (or appreciated) against the dollar because a pound now buys more dollars. The US investor in UK assets benefits from the depreciation of the dollar against the pound because on converting pounds back into dollars, he realizes a gain. By contrast, if the exchange rate one week later is \$1.55/£, it indicates that the dollar has appreciated against the pound because it takes fewer dollars to purchase one pound.

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²⁵ The different order of currency symbols in \$:€ is not an error but a difference in quotation style; it can be read as "one dollar is priced at 0.68493 euros."

PRACTICE PROBLEMS

The following information relates to Questions 1 and 2

A 12 percent coupon bond was issued on 1 February 2011. The clean price for the bond was \$930 on 31 March 2011 and \$980 on 30 June 2011. The coupon is paid semi-annually.

- 1 Which of the following is *closest* to the price return from 31 March 2011 to 30 June 2011?
 - A 5.26%
 - B 5.38%
 - C 8.60%
 - 2 Which of the following is *closest* to the total return from 31 March to 30 June 2011?
 - A 7.37%
 - B 8.42%
 - C 8.60%
-
- 3 A stock is held from 31 March 2011 to 30 June 2011. The 31 March 2011 stock price was \$65 and the 30 June 2011 price was \$83. The stock paid a \$1.80 dividend on 1 February. On 15 June, the company announced that the same amount of dividend would be paid on 1 August. The ex-dividend dates are a week before pay date. Which of the following is *closest* to the total return for the equity investment from 31 March to 30 June?
 - A 27.69%
 - B 29.54%
 - C 30.46%
 - 4 The value at the beginning of the year for a total return index was \$1,213.47, and the end-of-year value was \$1,336.42. During the year, dividend income of \$7.92 was received and \$5.24 of interest income was received. The value for the price index was \$911.01 at the beginning of the year and \$989.03 at the end of the year. If the gain from reinvested income is included in the income return, which of the following is *closest* to the income return for the index for the year?
 - A 1.08%
 - B 1.44%
 - C 1.57%
 - 5 A 12% coupon bond was purchased on 1 June 2016. The clean price of the bond on the purchase date is 100. The bond pays annual coupons on 1 March. On 1 December 2016, the clean price of the bond is 102. The total return of the bond for the six-month period is *closest* to:
 - A 7.8%.
 - B 8.0%.

- C 10.7%.
- 6 The total return for a bond is positive for a given period that lies between two interest payment dates. Given this information one:
- A can conclude that the clean price has increased.
 B can conclude that the clean price has decreased.
 C cannot conclude whether the clean price has increased or decreased.
- 7 A 6 percent coupon bond was purchased for \$1,000 at par when issued. The bond was sold four months later at a clean price of \$1,050. Assuming that the coupon is paid semi-annually, which of the following is *closest* to the total return for the four-month holding period?
- A 5.0%
 B 7.0%
 C 8.0%
- 8 A portfolio contains US dollars (\$), UK pound sterling (£), and Japanese yen (¥) equities. The investor's base currency is the Japanese yen. The value of the equities in millions and the value of the currencies are shown in the following table for 31 January and 28 February of a given year.

	31 Jan	28 Feb
US equities	\$110.20	\$98.50
UK equities	£123.00	£135.00
Japanese equities	¥1,678.00	¥1,801.00
One US dollar buys this many euros:	0.8	0.9
One euro buys this many UK pounds sterling:	0.67	0.71
One Japanese yen buys this many euros:	0.008811	0.009126

Which of the following is *closest* to the portfolio's base-currency return for the month?

- A -0.5%
 B 3.0%
 C 6.7%

The following information relates to Questions 9–11

An Italian investor owns 120,000 shares of a Swiss stock. The investor's base currency is the euro (€). The value of the stock in Swiss francs (CHF) and the currency values in the table below are for the beginning and end of a given year. The dividends were received at the end of the year.

	1 January	31 December
Share price	CHF 46.70	CHF 51.90
Dividend per share		CHF 2.30
€/CHF =	0.819672	0.862069

- 9 Which of the following is *closest* to the currency return that was made on the initial investment for this year?
- A -4.9%

- B 4.9%
- C 5.2%
- 10 Which of the following is *closest* to the base-currency return for the year?
- A 10.4%
- B 21.2%
- C 22.1%
- 11 Which of the following is *closest* to the return from the interaction of the local return and the currency return for the year?
- A -0.8%
- B 0.7%
- C 0.8%

- 12 A UK investor held equity positions in US, Brazilian, and Japanese equities. The investor's base currency is the British pound (£). The equity returns for the holding period in the local currencies (LC) are shown below. The changes in the US dollar (\$), the Brazilian real (R\$), and the Japanese yen (¥), relative to the British pound, are also shown.

LC	Equity return in LC	Change in currency value relative to British pound
US	5%	-15%
Brazil	-4%	-6%
Japan	-12%	2%

- The investor's base-currency return is greatest for the:
- A US equity investment.
- B Brazilian equity investment.
- C Japanese equity investment.
- 13 An investor purchased 3,000 shares of stock at a price of \$56. To purchase the stock, the investor financed 40 percent of the acquisition with borrowed funds. A month later, a dividend of \$0.80 per share was received and the stock was sold at a price of \$61. Trading expenses are \$30 on the purchase and \$30 on the sale. The interest on margin funds was 2 percent annually. Which of the following is *closest* to the monthly gross-of-fee return for the investment?
- A 10.4%
- B 17.1%
- C 17.3%
- 14 An institutional investor has the long and short positions in the table below for one year.

Long Position	
Beginning Share Value	\$410.00
Dividend	\$3.20
Price Change	9.4%

Short Position	
Beginning Share Value	\$560.00
Dividend	\$1.20
Price Change	18.2%

The investor pays a fee of 0.75 percent times the beginning short position to borrow the shares. Assume this fee and the dividends occur at the end of the year and that the short position does not require margin. Which of the following is *closest* to the annual gross-of-fee return for this investment?

- A -16.0%
 - B -15.4%
 - C -6.8%
- 15 The 2011 values for a pension fund are provided in the table below, in millions of dollars.

Pension Assets	
Beginning value on 1 January 2011	\$50.11
Change in value in 2011	6.3%

Pension Liabilities	
Beginning value on 1 January 2011	\$47.23
Change in value in 2011	8.4%

Which of the following is *closest* to the surplus return for this fund?

- A -28.1%
 - B -2.1%
 - C 7.3%
- 16 A portfolio manager buys a stock and shortly thereafter sells it. The table below provides the sequence of events and stock prices associated with the transaction.

Date	Closing Stock	
	Price	Event
2 May 2011	\$45.12	Purchase order is executed
3 May 2011	\$45.57	
4 May 2011	\$46.16	Sale order is executed
5 May 2011	\$46.91	Cash is exchanged for stock
6 May 2011	\$47.00	
9 May 2011	\$47.68	Stock is exchanged for cash

Which of the following is *closest* to the holding period return?

- A 1.6%
 - B 2.3%
 - C 5.7%
- 17 A portfolio manager purchases 1,000 shares of stock in March 2011 and sells them later in the month. The purchase and the sale each incur a \$50 commission. The table below provides the sequence of events and stock prices associated with the transaction.

Date	Closing Stock Price	Event
7 March 2011	\$21.39	Trade date of purchase
8 March 2011	\$21.64	Ex-dividend date
9 March 2011	\$22.02	
10 March 2011	\$21.74	Settlement date of purchase
11 March 2011	\$21.53	
14 March 2011	\$21.36	Trade date of sale
15 March 2011	\$21.40	
16 March 2011	\$21.61	
17 March 2011	\$21.41	Settlement date of sale
18 March 2011	\$21.19	
5 April 2011	\$20.70	\$0.80 per share dividend paid

Which of the following is *closest* to the holding period return?

- A -0.6%
 - B 1.7%
 - C 3.1%
- 18 An investor owns a stock during 2011. Inflation rose during 2011 as reflected in the Consumer Price Index (CPI). A dividend is received at the end of the year. The table below provides beginning- and end-of-year stock prices and CPI values.

	1 January 2011	31 December 2011
Stock price	\$45.24	\$38.56
Dividend paid		\$2.55
CPI values	155.8	159.1

Which of the following is *closest* to the exact real return (the geometric difference between the nominal holding period return and the inflation rate) for the year?

- A -11.25%
- B -11.02%
- C -7.20%

The following information relates to Questions 19 and 20

An initial investment of \$2,500 in Fund A or Fund B generates a return of 9.5 percent per year. The tax rate applied to investment returns is 32 percent. An investment in Fund A would result in the realization of a tax on gains every year. An investment in Fund B defers the tax on gains until the investment is sold.

- 19 Over a one-year time horizon, which of the following is *closest* to the difference in post-tax percentage returns for an investment in Fund A versus that in Fund B?
- A 3.04%
 - B 6.46%

- C 32.00%
- 20 Assume both investments are liquidated in 10 years and that a tax is paid on Fund B's gain at that time. Which of the following is *closest* to the difference in post-tax annual percentage returns for an investment in Fund A versus that in Fund B?
- A 0.75%
- B 1.11%
- C 3.04%
-
- 21 The portfolio manager for a pension fund incurs quarterly trading expenses of 0.20 percent. The annual investment management fee for the portfolio is 1.60 percent and is paid on a quarterly basis. The portfolio's quarterly gross-of-fees return is 5.0 percent. The pension fund sponsor (the employer) pays the fund's investment management fees. Which of the following is *closest* to the portfolio's quarterly net-of-fees return that should be used to evaluate the portfolio manager?
- A 4.4%
- B 4.6%
- C 5.0%
- 22 An investor purchased a 7 percent coupon bond at par for \$1,000.00 immediately after the payment of a coupon and held it 18 months. Coupons were received semi-annually and reinvested at the stated annual rate of 4 percent compounded semi-annually. The bond price when sold was \$950.41. Which of the following is *closest* to the effective annual rate of return from the bond in the investor's portfolio?
- A 3.7%
- B 3.8%
- C 5.8%
- 23 A fund has quarterly returns of -9.6% , 4.4% , and -4.7% for the first three quarters of the year. Which of the following is *closest* to the fourth quarter return that is required to restore the fund to its beginning-of-year value?
- A 3.3%
- B 9.9%
- C 11.2%
- 24 A fund lost half its nominal value in the first year. Which of the following is *closest* to the annual nominal rate of return required over the next four years to restore the fund to its beginning real value? The inflation rate is 4 percent annually for all five years.
- A 18.9%
- B 20.1%
- C 24.9%
- 25 An asset priced at \$19 increases in price to \$21 over one year. Which of the following is *closest* to the continuously compounded rate of return for the asset?
- A 10.0%
- B 10.5%
- C 11.1%
- 26 An analyst observes the following year-end stock prices:

Year	Year-end stock price
1	\$9.14
2	\$9.23
3	\$8.88
4	\$8.93
5	\$9.22
6	\$9.99

Which of the following is *closest* to the compound annual rate of growth?

- A 1.49%
- B 1.79%
- C 1.87%

The following information relates to Questions 27 and 28

Joan Bradham is a performance analyst for US-based Garnet Portfolio Management. While reviewing the historical performance of the firm's equity investments, she makes the following statements.

- Statement #1 "If a series of equal contributions are made at the beginning of each year, the arithmetic mean is preferred over the geometric mean because arithmetic mean will provide the best representation of the investor's terminal wealth."
- Statement #2 "Relative to the geometric mean, the arithmetic mean represents a more conservative estimate of an investment's average return because it is generally lower."

Turning her attention to the performance of the firm's money market fund, Bradham makes the following statements.

- Statement #3 "The fund's most recent quarterly performance was quite impressive. The fund had a return of 3.2 percent, strongly outperforming its benchmark. I will report the annual performance of the money market fund to our investors as 13.4 percent, which I calculate as $1.032^4 - 1$."
- Statement #4 "I have been examining the historical performance of the money market fund relative to global macroeconomic conditions. In the statistical analysis, I am predicting the arithmetic mean rather than the geometric mean because it is best suited for this usage."

27 Are Statement #1 and Statement #2 correct?

	Statement #1	Statement #2
A	No	No
B	No	Yes
C	Yes	Yes

28 Are Statement #3 and Statement #4 correct?

	Statement #3	Statement #4
A	No	No
B	No	Yes
C	Yes	Yes

- 29 For a fund that uses unit value pricing and has intraperiod cash flows, the net asset value (NAV) of a unit is:
- A** calculated before deducting fund liabilities.
 - B** used to calculate money-weighted rates of return.
 - C** not affected by capital withdrawals from the fund.

The following information relates to Questions 30–32

John McMaan has just received from his portfolio manager a summary report on his investment returns for the 15-year period from 1 January 1996 to 31 December 2010. The benchmark given to the portfolio manager by McMaan has always been inflation.

Data on John McMaan's investment returns and inflation benchmark for the years 1996 through 2010

	Arithmetic averages		Geometric averages	
	Inflation	Portfolio return (nominal)	Inflation	Portfolio return (nominal)
1/1/1996 to 31/12/2010 (15 years)	4.4%	6.8%	4.0%	6.0%
1/1/2001 to 31/12/2010 (10 years)	3.8%	5.0%	3.0%	4.0%
1/1/2006 to 31/12/2010 (5 years)	2.0%	0.6%	1.5%	0.5%

- 30 The average annual real rate of return for the five-year period from 1 January 2001 to 31 December 2005, in analyzing historical portfolio returns, is *closest* to
- A** 2.6%.
 - B** 3.0%.
 - C** 3.8%.
- 31 Suppose that \$1,000 was invested in McMaan's portfolio on 1 January 2006 when it could have bought 50 units of a given good. Suppose the price of that good increased with inflation from 1 January 2006 to 31 December 2010. The number of units of the same good that McMaan could have bought on 31 December 2010 with the accumulated value of the \$1,000 is *closest* to (rounded down to a whole number of units)

- A 46.
 B 47.
 C 52.
- 32 The highest variability of McMaan's portfolio returns, as measured by the variance, occurred in the period
- A 1996–2000.
 B 2001–2005.
 C 2006–2010.

The following information relates to Questions 33–35

A 12 percent coupon bond is bought at par on 1 January 2011, the time of issuance. Semi-annual coupon payments are reinvested at market rates. The bond's quarterly value and the value of the portfolio (inclusive of reinvested coupons) are shown in the table below for an 18-month period.

	Bond Value	Coupons Paid	Total Value of Portfolio
1 January 2011	\$1,000.00		\$1,000.00
1 April 2011	\$1,005.12		\$1,005.12
1 July 2011	\$1,022.11	\$60.00	\$1,082.11
1 October 2011	\$992.29		\$1,053.04
1 January 2012	\$978.92	\$60.00	\$1,100.42
1 April 2012	\$966.23		\$1,089.24
1 July 2012	\$933.90	\$60.00	\$1,118.44

- 33 Which of the following is *closest* to the portfolio's annualized time-weighted rate of return for the 18-month period?
- A 7.7%
 B 11.8%
 C 18.3%
- 34 Assume that the coupons for the bond in the previous question are not reinvested but are withdrawn by the investor. Which of the following is *closest* to the portfolio's time-weighted rate of return?
- A 1.9%
 B 2.8%
 C 7.6%
- 35 Using the same assumption—that the coupons for the bond are not reinvested but are withdrawn—which of the following is *closest* to the portfolio's money-weighted rate of return?
- A -8.4%

- B 3.8%
 - C 7.9%
-

36 Consider the use of the money-weighted return and time-weighted return for each of the following scenarios:

- Scenario A A portfolio manager manages a portfolio of illiquid microcap stocks and hedge fund investments for a trust. Although the trust requires annual, predetermined disbursements from the fund, the manager controls investment choices and the timing of the liquidation of investments to meet the disbursements. The manager prepares monthly valuations for the trustees.
- Scenario B A portfolio of publicly traded large-cap stocks and commodity futures for a client has performed quite well over time. The client's contributions, however, have been quite variable. In the fall of 2007, the client required a large disbursement for a divorce settlement. Contributions have been lumpy in nature because of the client's performance-based executive compensation.
- Scenario C A portfolio manager is preparing a presentation for a long-term client where the portfolio has underperformed its benchmarks, partly because the client had made a large, one-time contribution from an inheritance to the fund during a prolonged bear market. The client is considering a switch to another manager and wishes to know how much a dollar invested in the current fund at inception would be worth as of the most recent quarter.

In which of the scenarios would the money-weighted return be the *most* appropriate measure of the portfolio manager's performance?

- A Scenario A
- B Scenario B
- C Scenario C

The following information relates to Questions 37 and 38

An investment account earns the following dividends and receives a contribution from the investor during 2010. Dividends are not reinvested in the account but, rather, are paid out to the investor. Account values reflect contributions and are after the payment of dividends.

	Dividends	Contributions	Account Value
1 January 2010			\$237,000
1 July 2010	\$8,000		\$188,000
1 October 2010		\$40,000	\$220,000
31 December 2010	\$8,000		\$329,000

- 37 Which of the following is *closest* to the account's time-weighted rate of return for 2010?
- A 13.6%
- B 21.3%
- C 48.2%
- 38 Which of the following is *closest* to the account's money-weighted rate of return using the IRR formula for 2010?
- A 8.5%
- B 21.1%
- C 28.1%

The following information relates to Questions 39 and 40

A large manufacturer offers its employees a pension plan but has not paid retirement benefits yet. In April 2010, the company made a contribution to the plan. In October of that year, following a partial windup of the business, a large number of employees withdrew the value of their accumulated benefits. The pension fund earns quarterly dividends that are reinvested in the fund. The pension fund values shown in the table below reflect dividends, withdrawals by employees, and sponsor contributions (in thousands).

	Dividends	Withdrawals	Contributions	Fund Value
1 January 2010	\$2,000			\$93,200
1 April 2010	\$2,000		\$13,000	\$111,300
1 July 2010	\$2,000			\$67,900
1 October 2010	\$2,000	\$21,000		\$49,900
31 December 2010				\$74,100

- 39 Which of the following is *closest* to the fund's time-weighted rate of return for 2010?
- A -48.57%
- B -29.78%
- C -0.22%
- 40 Which of the following is *closest* to the fund's money-weighted return using the modified Dietz return formula for 2010?
- A -31.2%

- B -11.4%
- C 2.2%

The following information relates to Questions 41 and 42

A mutual fund reports the following data:

- A fund's number of units at the end of August is 10 million.
- On 4 September a withdrawal is made of \$8.3 million.
- On 22 September a contribution of \$6.7 million is made.
- Market values in the table below are before portfolio transactions of the same date.

Additional data (all in millions except for NAV) are provided in the table. Data for the added units on 22 September and the units before flow on 30 September are purposely omitted.

Date	Market Value	Units before Flow	NAV	External Cash Flow	Added Units
31 August	\$45.80	10.000	4.580		0.000
4 September	\$42.30	10.000	4.230	(\$8.30)	-1.962
13 September	\$37.70	8.038	4.690		0.000
22 September	\$43.60	8.038	5.424	\$6.70	?
30 September	\$47.50	?	5.122		0.000

- 41 Which of the following is *closest* to the added units on 22 September?
- A 0.810
 - B 0.834
 - C 1.235
- 42 Which of the following is *closest* to the time-weighted rate of return for September?
- A 2.44%
 - B 3.71%
 - C 11.83%

The following information relates to Questions 43–45

The following table provides information for a portfolio composed of two assets, Asset A and Asset B, for the month of December. Contributions and withdrawals occur at the end of the day.

	Date (t)	Asset A	Asset B
Market Value	30 Nov	\$241.00	\$88.00
Contribution	3 Dec	\$34.00	\$12.00
Withdrawal	22 Dec	\$14.00	\$9.00
Market Value	31 Dec	\$289.00	\$86.00

- 43 Given the figures in the table, which of the following is *closest* to the IRR for December for Asset A?
- A 3.5%
- B 8.3%
- C 10.5%
- 44 Given the figures in the table, which of the following is *closest* to the Modified Dietz return for December for Asset B?
- A -5.49%
- B -5.20%
- C 1.25%
- 45 In this example, the modified Dietz return for the entire portfolio will be consistent with the modified Dietz return for the portfolio segments when the segment weightings are based on
- A end-of-period values.
- B beginning-of-period values.
- C beginning-of-period values adjusted for external cash flows.

The following information relates to Questions 46–50

Stephan Ryan began investing money in a registered education saving plan (RESP) for two of his grandchildren, Laura and Noah, in 2009 and for his third grandchild, Sophia, in 2010. Ryan has decided that the investments would be initially in bonds in Laura's account, in stocks in Noah's account, and in real estate investment trusts in Sophia's account.

- 46 Financial data for the individual accounts for March and April 2009 are shown below. During that period, there was no purchase or sale of securities in any of the accounts and no contribution or withdrawal.

Data on the individual accounts of the RESP for March and April 2009 (in thousands of dollars except for returns)

	Laura	Noah
Market value on 1 March 2009	80.0	20.0
Coupons received on 15 March 2009 and reinvested in Laura's portfolio	3.2	—
Dividends received on 15 March 2009 and held in cash in Noah's portfolio	—	0.6

(continued)

(Continued)

	Laura	Noah
Total return from 1 March 2009 to 30 April 2009	5%	not available
Price return from 1 March 2009 to 30 April 2009	not available	-1%

The total return of the RESP for the period from 1 March 2009 to 30 April 2009 was equal to:

- A 3.8%.
- B 4.4%.
- C 4.6%.

- 47 Data for the individual accounts for the year 2010 are shown below (amounts in thousands of dollars):

	Laura	Noah	Sophia	Total RESP Fund
Market value on 1 January	100.0	40.0	0	140.0
New contributions made on 31 January	10.0	20.0	30.0	60.0
Market value 31 December	105.0	65.0	31.0	201.0

If the method used is to weight the modified Dietz return of the individual accounts with the weights of those accounts at the beginning of the year, the RESP fund return in 2010 is *closest* to:

- A -0.82%.
 - B -0.11%.
 - C 0.00%.
- 48 Based on the same data as in the previous question, if one weights the modified Dietz returns of the individual accounts with the adjusted beginning values (i.e., the weights of those accounts at the beginning of the year adjusted for the new contributions made during the year), the RESP fund return in 2010 is *closest* to:
- A 0.51%.
 - B 0.69%.
 - C 1.18%.
- 49 Ryan is aware that there were large variations of returns in the RESP accounts during the year and decides to analyze the 2010 results using time-weighted rates of return. For that purpose he adds to his data the values of the accounts immediately before the new contributions were made. The complete financial data are presented below:

Data on Individual Accounts of the RESP for the year 2010 (amounts in thousands of dollars)

	Laura	Noah	Sophia	Total RESP Fund
Market value on 1 January	100.0	40.0	0	140.0
Market value on 31 January before new contributions	80.0	45.0	0	125.0

(Continued)

	Laura	Noah	Sophia	Total RESP Fund
New contributions made on 31 January	10.0	20.0	30.0	60.0
Market value 31 December	105.0	65.0	31.0	201.0

The contribution of Sophia's account return to the total RESP fund return for the period from 31 January to 31 December 2010 is *closest* to:

- A 0.51%.
 - B 0.54%.
 - C 3.06%.
- 50 Based on the data in Question 50, Ryan intends to write the following in his summary report on the RESP fund:
- "For the period from 1 January 2010 to 31 December 2010, the rate of return
- in Laura's account, that best reflects the influence of the timing of external cash flows, is below -4 percent." (**Statement 1**)
 - "in Noah's account, that best reflects the investment management of the assets irrespective of the timing of external cash flows, is above 12 percent." (**Statement 2**)
 - "in Sophia's account, that best reflects the investment management of the assets irrespective of the timing of external cash flows but still reflects the fact that money was deposited for the first time only on 31 January, is above 3.5 percent." (**Statement 3**)

Which of Ryan's statements is *incorrect*?

- A Statement 1
- B Statement 2
- C Statement 3

SOLUTIONS

- 1 A is correct. The semi-annual coupon is $12\%/2 \times \$1,000 = \60 . There are two months between 1 February and 31 March. The accrued interest from 1 February to 31 March is $\$60 \times 2/6 = \20 . If the bond was purchased on 31 March, the bond investor would pay \$20 in accrued interest. The dirty price (total price) paid on 31 March would be $\$930 + \$20 = \$950$. The price return is the change in the clean prices divided by the dirty price:

$$\text{Price return} = \frac{\$980 - \$930}{\$950} = 0.0526 = 5.26\%$$

B is incorrect and is calculated by using the clean price in the denominator. C is incorrect and is calculated by using the clean price in the denominator and dirty prices in the numerator.

- 2 B is correct. From the previous question, the bond investor would pay \$20 in accrued interest on 31 March. There are five months between 1 February and 30 June. The accrued interest from 1 February to 30 June is $\$60 \times 5/6 = \50 . If the bond is sold on 30 June, the bond investor receives \$50 in accrued interest. So, the investor would pay \$20 but would later receive \$50; therefore, the investor would net \$30 in interest. Thus, the change in accrued interest between 31 March and 30 June is $\$50 - \$20 = \$30$. The income return is the change in accrued interest divided by the dirty price paid: $\$30/\$950 = 3.16\%$.

Using the answer from the previous question, the total return is the sum of the income and price return: $3.16\% + 5.26\% = 8.42\%$. Or it can be shown as

$$\text{Total return} = \frac{\$980 - \$930 + \$30}{\$950} = 0.0842 = 8.42\%$$

Alternatively, we can calculate the total return as the percentage change in dirty prices. The dirty price on 30 June is $\$50 + \$980 = \$1,030$.

$$\text{Total return} = \frac{\$1,030 - \$950}{\$950} = 0.0842 = 8.42\%$$

A is incorrect and is calculated by using two months of interest in the calculation of the income return. C is incorrect and is calculated by using the clean prices in the calculations of price and income return.

- 3 A is correct. There is no income return here because a dividend was not received. The investor purchased the stock after the January ex-dividend date and sold it before the July ex-dividend date. Note that there is no accrual of the dividend amount as there is for bond coupons. When investing in bonds, there is accrual of income between coupon payment dates and that accrual of income is paid to the seller from the buyer; the clean price, also called the quoted price, does not include that accrual of income. For stocks the present value of any expected dividend is included in the quoted stock price. So, the total return is equal to the price return because the income return is zero:

$$\text{Total return} = \frac{V_1 - V_0 + D}{V_0} = \frac{\$83 - \$65 + \$0}{\$65} = \frac{\$18.00}{\$65} = 0.2769 = 27.69\%$$

B is incorrect because four months of dividends are included in the income return component. C is incorrect because six months of dividends are included in the income return component.

- 4 C is correct. If the gain from reinvested income is included in the income return, we will calculate the total return for the index and subtract the price return to arrive at the income return. This question illustrates Case 1 in Section 2.1 of the text. The total return is calculated as

$$\begin{aligned} \text{Total return} \\ &= \frac{V_1 - V_0}{V_0} = \frac{\$1,336.42 - \$1,213.47}{\$1,213.47} = \frac{\$122.95}{\$1,213.47} = 0.1013 = 10.13\% \end{aligned}$$

The price return is exclusive (x) of the gain from reinvested income and is

$$\text{Price return} = \frac{V_1^x - V_0^x}{V_0^x} = \frac{\$989.03 - \$911.01}{\$911.01} = \frac{\$78.02}{\$911.01} = 0.0856 = 8.56\%$$

The income return includes the gain from reinvested income and is the difference between the total return and the price return: $10.13\% - 8.56\% = 1.57\%$.

A is incorrect and is calculated as the sum of the dividend and interest income divided by the beginning-of-year total return index value:

$$\frac{D}{V_0} = \frac{\$7.92 + \$5.24}{\$1,213.47} = 0.0108 = 1.08\%$$

This response illustrates including gains from reinvested income in the price return, not the income return as specified. B is incorrect and is calculated as the sum of the dividend and interest income divided by the beginning-of-year price index value.

- 5 A is correct. The total return for a bond purchased and sold within a single period is calculated based on the change in the clean price plus any change in accrued interest. The change in the clean price was 2. On 1 June (purchase date), the bond had accrued interest of 3 = $100 \times 12\% \times (3/12)$. As of 1 December (sell date), the bond had accrued interest of 9 = $100 \times 12\% \times (9/12)$. The change in accrued interest is $9 - 3 = 6$.

Total return is calculated as follows:

$$\text{Total return} = (\text{Ending clean price} - \text{Beginning clean price} + \text{Change in accrued interest}) / (\text{Beginning dirty price}),$$

$$\text{Total return} = (102 - 100 + 6) / (100 + 3),$$

$$\text{Total return} = 7.77\%.$$

- 6 C is correct. From the information given, one cannot establish whether the clean price increased or decreased. The total return of a bond incorporates both the change in the clean price and the change in accrued interest. It is possible to have a positive total return along with a decrease in the clean price if the change in accrued interest is positive and larger than the decrease in the clean price.
- 7 B is correct. The semi-annual coupon is $6\%/2 \times \$1,000 = \30 . The accrued interest over four months is $\$30 \times 4/6 = \20 . When the bond is sold at the end of the four-month holding period, the investor receives \$20 in income. Because the bond was bought at par and is sold at the clean price of \$1,050, the holding period return is

$$\frac{\$1,050 - \$1,000 + \$20}{\$1,000} = 0.07 = 7.0\%$$

A is incorrect and omits the accrued interest. C is incorrect and is calculated by using six months of accrued interest.

- 8 A is correct. It will be easiest to convert all the portfolio values and corresponding returns to a common currency. We'll use the euro (the common currency in all the exchange rates is the euro) and then convert the euro values to yen values. First, let's restate the exchange rates in the table. For the beginning values, we have €0.80/\$, £0.67/€, and €0.008811/¥, respectively. For the ending values, we have €0.90/\$, €0.71/£, and €0.009126/¥, respectively. Then, converting the beginning and ending values to euros, we have

$$\text{Beginning Value: } (\$110.2 \times \text{€}0.80/\text{\$}) + (\text{£}123.0 \times \text{€}/\text{£}0.67) + (\text{¥}1,678.0 \times \text{€}0.008811/\text{¥}) = \text{€}286.5269$$

$$\text{Ending Value: } (\text{\$}98.5 \times \text{€}0.90/\text{\$}) + (\text{£}135.0 \times \text{€}/\text{£}0.71) + (\text{¥}1,801.0 \times \text{€}0.009126/\text{¥}) = \text{€}295.2268$$

Note that for each calculation, we simply set up the formula so the other currencies cancel and we are left a value in euros. Note also that the pound–euro calculation uses an exchange rate stated in terms of the euro. So, we must invert this rate in the calculation. The fund's return in euros is then $\text{€}295.2268/\text{€}286.5269 - 1 = 3.0363\%$.

To convert this to a yen return, first calculate the euro appreciation/depreciation relative to the yen. Because we are interested in the euro change in value, we put that currency in the denominator of a percentage change calculation:

$$\frac{\text{¥}/\text{€}0.009126}{\text{¥}/\text{€}0.008811} - 1 = -3.4517\%$$

The return in yen is then the compounding of the equity return in euros and the euro change in value: $(1.030363) \times (1 - 0.034517) - 1 = -0.52\%$.

You can practice by calculating the returns in the dollar and pound. The change in the value of the euro relative to the dollar is

$$\frac{\text{\$/€}0.9}{\text{\$/€}0.8} - 1 = -11.11\%$$

The change in the value of dollar equity is $(1.030363) \times (1 - 0.1111) - 1 = -8.41\%$.

The change in the value of the euro relative to the pound is (again putting the euro in the denominator because it is its change we are interested in)

$$\frac{\text{£}0.71/\text{€}}{\text{£}0.67/\text{€}} - 1 = 5.9701\%$$

The change in the value of pound equity is $(1.030363) \times (1.059701) - 1 = 9.19\%$.

Another way of calculating the return in yen is to compare the beginning and ending values of the portfolio in yen. The beginning value in yen using the 31 January exchange rates is equal to

$$\begin{aligned} & 110.20 \text{ dollars} \times \frac{1}{0.008811} \left(\frac{\text{yen}}{\text{euro}} \right) \times 0.8 \left(\frac{\text{euro}}{\text{dollar}} \right) \\ & + 123.00 \text{ pounds} \times \frac{1}{0.008811} \left(\frac{\text{yen}}{\text{euro}} \right) \times \frac{1}{0.67} \left(\frac{\text{euro}}{\text{pound}} \right) \\ & + 1,678 \text{ yen} \\ & = 10,005.67 \text{ yen} + 20,835.56 \text{ yen} + 1,678 \text{ yen} \\ & = 32,519.23 \text{ yen} \end{aligned}$$

The ending value in yen, using the 28 February exchange rates, could be calculated in a similar manner and is equal to 32,350.07. So, the February return in yen is equal to

$$\frac{32,350.07 - 32,519.23}{32,519.23} = -0.52\%$$

B is incorrect because it is the return in euros. C is incorrect because it inverts the currency return.

- 9 C is correct. The currency return is defined as $R^C = (S_1^{d/f} / S_0^{d/f}) - 1$.

$$R^C = (0.862069 / 0.819672) - 1 = 5.17\%$$

CHF has strengthened against the euro because at year-end it takes fewer CHF to buy a euro.

A is incorrect because it inverts the currency return. B is incorrect because it uses the currency gain divided by the ending value (not the beginning value).

- 10 C is correct. The most direct method of calculating the return in the base currency (€) is to calculate the local return in Swiss francs and the currency change in value. The local return in Swiss francs factors in the capital gain and dividend:

$$\frac{\text{CHF}51.90 + 2.30}{\text{CHF}46.70} - 1 = 16.06\%$$

The base return (€) is then the compounded return of the holding period return in Swiss francs and the currency return (already calculated as 5.17%): $(1.1606)(1.0517) - 1 = 22.06\%$.

A is incorrect because it inverts the currency return. B is incorrect because it does not use the entire currency return component (i.e., it does not include the interaction term).

- 11 C is correct.

The interaction return is the third term of Equation 8b in the reading:

$$R = R^{FC} + R^C + R^{FC}R^C$$

where

R = base-currency return

R^{FC} = local return

R^C = currency return

$R^{FC} \times R^C$ = return due to the interaction of the local return and the currency return

$$\text{Here, } R^{FC}R^C = \left[\frac{51.90 + 2.30}{46.70} - 1 \right] \left[\frac{0.862069}{0.819672} - 1 \right] = 0.83\%$$

A is incorrect because it inverts the currency return. B is incorrect because it divides the interaction gain by the ending value (not the beginning value).

- 12 B is correct. The investor's base-currency return will be a function of the change in the currency value (currency return), the equity return in local currency (local return), and the interaction between the two. This is shown in Equation 8:

$$\begin{aligned} R &= (1 + R^C) \times (1 + R^{FC}) - 1 \\ &= R^C + R^{FC} + (R^C \times R^{FC}) \end{aligned}$$

In all three countries, the simple sum of the first two components is -10% (e.g., Japan: $-12\% + 2\% = -10\%$). However, a simple sum of returns does not factor in the third term, the interaction between the equity local return and the currency

return. Notice that the third term is positive for only the Brazilian market because it is the product of two negative values. Therefore, the Brazilian market will have a base-currency return slightly higher than -10% , whereas the US and Japanese markets will have returns slightly less than -10% . You can verify that the Brazilian market will provide the highest base-currency return:

$$\text{United States: } R = (1 - 0.15)(1 + 0.05) - 1 = -10.75\%$$

$$\text{Brazil: } R = (1 - 0.06)(1 - 0.04) - 1 = -9.76\%$$

$$\text{Japan: } R = (1 + 0.02)(1 - 0.12) - 1 = -10.24\%$$

The intuition as to why the Brazilian market's interaction term will be positive and slightly boost the return (even though both equity and currency returns are negative) can be understood from the text decomposition of returns into areas. In this example for Brazil, the calculation for Area C, the currency change in value, assumes that the beginning equity value is multiplied by a currency depreciation. However, the ending value for the Brazilian equity is actually what will be converted to British pounds. Because the ending equity value is smaller than the beginning value, the calculation for Area C overestimates the loss from the currency depreciation. The calculation for Area D, the interaction term, then corrects the overestimation.

- 13** B is correct. The total purchase price is $3,000 \times \$56 = \$168,000$. The investor must put up equity of $60\% \times \$168,000 = \$100,800$. The remaining $\$67,200$ is borrowed. Including the trading expenses on the stock purchase, the total initial equity investment is $\$100,830$. In one month, the sales price is $3,000 \times \$61 = \$183,000$ and the capital gain is $\$183,000 - \$168,000 = \$15,000$. Dividends received are $3,000 \times \$0.80 = \$2,400$. Interest paid for the month is $\$67,200 \times 2\%/12 = \112 .

Including the expenses on the sale, the gain on the transaction in one month is $\$15,000 + \$2,400 - \$112 - \$30 = \$17,258$. Subtracting the trading expenses on the purchase of $\$30$ results in a total gain of $\$17,228$. The return on the equity investment is $\$17,228/\$100,830 = 17.1\%$.

Using the formula in the text, we would first estimate the cash basis return:

$$\begin{aligned} &= (\text{Capital gain} + \text{Dividends} - \text{Trading expenses})/\text{Beginning investment} \\ &= (\$15,000 + \$2,400 - \$30 - \$30)/\$168,030 = 10.32\% \end{aligned}$$

The leverage ratio, L , equals 0.40 because the equity is 60% . Using the text formula,¹

$$R_{LE} = \frac{R_{CB} - iL}{1 - L} = \frac{0.1032 - 0.02/12(0.4)}{1 - 0.40} = \frac{0.1025}{0.60} = 0.1709 = 17.1\%$$

A is incorrect because it assumes leverage is not used and omits trading costs. C is incorrect because it assumes leverage is used but at no interest cost. It also omits trading costs.

- 14** A is correct. The long-position price appreciation is $\$410 \times 9.4\% = \38.54 . Adding the dividend provides a profit of $\$41.74$. The stock short increases in price, so the investor suffers a loss of $\$560 \times 18.2\% = \101.92 . The institutional investor owes a dividend of $\$1.20$ to the stock owner. The borrowing cost is $\$560 \times 0.75\% = \4.20 . The loss on the short position is then $-\$101.92 - \$1.20 - \$4.20 = -\107.32 .

¹ If you carry this calculation out several decimal places, the 17.1% calculated using the text formula is not exactly the same as the 17.1% calculated previously. The latter return is 0.002% higher because the trading expenses increase the initial investment and are not margined.

The initial investment is the cost of the long position because the short position does not require margin in this problem. The total return on both the long and short is $(\$41.74 - \$107.32)/\$410 = -16.0\%$. Note that the gain on the long position is outweighed by the loss on the short position. In a long position, the gains are unlimited because there is no limit on asset price appreciation. Losses are limited to the original investment amount. In a short position, however, gains are limited to 100 percent (the minimum stock price is zero) and losses are unlimited (the stock price upside is theoretically unlimited).

B is incorrect because it assumes that the dividend in the short position is received (rather than paid). C is incorrect because it assumes that an initial investment in the short position must be made.

- 15** A is correct. The fund's beginning surplus was $\$50.11 - \$47.23 = \$2.88$. The pension assets have increased in value to $\$50.11 \times 1.063 = \53.27 . The pension liabilities have increased in value to $\$47.23 \times 1.084 = \51.20 . The new surplus is $\$53.27 - \$51.20 = \$2.07$. The decline in the surplus on a percentage basis is $\$2.07/\$2.88 - 1 = -28.1\%$.

B is incorrect because it is the return on the assets minus the return on the liabilities. C is incorrect because it is the return on a portfolio of assets and liabilities.

- 16** B is correct. 2 May is the trade date for the purchase, which is the appropriate date for accounting for the purchase. The stock price at which the order was executed on the trade date, \$45.12, is also the appropriate price for accounting. The settlement date for the purchase is 5 May, which is when the portfolio manager pays the \$45.12 cash for the stock.

The trade date for the sale is 4 May. The settlement date for the purchase is 9 May, which is when the portfolio manager receives \$46.16 in cash for the stock. The goal of performance measurement is to determine whether the manager adds value. The trade date and the price on that date best reflect the timing and cash flow implication of the manager's decision. No dividends were paid, so the holding period return is

$$\frac{V_1 - V_0 + D}{V_0} = \frac{\$46.16 - \$45.12 + 0}{\$45.12} = \frac{\$1.04}{\$45.12} = 0.0230 = 2.30\%$$

A is incorrect because it uses settlement dates. C is incorrect because it uses the first trade date and last settlement date prices.

- 17** C is correct. We will use the 7 March and 14 March trade date prices (\$21.39 and \$21.36) to determine the capital gain from the purchase and sale. We use trade date prices, not settlement date prices, for manager evaluation because the order will be executed at trade date prices. Even though the pay date is in April, the investor will receive the dividend because the stock was purchased on the record date. The trading day before the ex-dividend day (8 March) is the record date (7 March). The record date is the last day an investor can own the share and be entitled to receive the dividend. The holding period return should recognize the value of the dividend even though it had not yet been paid.

The investor bought and sold 1,000 shares and received an \$0.80 per share dividend. The commission was \$50 on the purchase and \$50 on the sale, so the holding period return is

$$= \frac{\$21,360 - \$21,390 + \$800 - \$50 - \$50}{\$21,390} = \frac{\$670}{\$21,390} = 0.0313 = 3.13\%$$

A is incorrect because it omits the dividends. B is incorrect because it uses settlement dates.

- 18 B is correct. The nominal holding period return is

$$\frac{V_1 - V_0 + D}{V_0} = \frac{\$38.56 - \$45.24 + \$2.55}{\$45.24} = \frac{-\$4.13}{\$45.24} = -0.0913 = -9.13\%$$

The inflation rate over the year is

$$IR = \frac{I_1}{I_0} - 1 = \frac{159.1}{155.8} - 1 = 2.12\%$$

To solve for the geometric difference (the real return calculated exactly) between the nominal holding period return and the inflation rate, we use

$$R^{real} = \frac{1 + R}{1 + IR} - 1 = \frac{1 - 0.0913}{1 + 0.0212} - 1 = -11.02\%$$

A is incorrect because -11.25% is the arithmetic difference (the approximate real return) between the nominal holding period return and the inflation rate. It is calculated as $-9.13\% - 2.12\% = -11.25\%$. Although the approximate real return is acceptable in many cases, especially when the difference between the exact and approximate real returns is small, the question asks for the geometric difference (exact calculation of the real return). C is incorrect and compounds the nominal return with the inflation rate instead of discounting it.

- 19 A is correct. The pre-tax future value for an investment in Fund A or B is $\$2,500 \times 1.095 = \$2,737.50$. Applying the tax to the $\$237.50$ gain in the case of Fund A, the after-tax future value = $\$2,500 + \$237.50 \times (1 - 0.32) = \$2,661.50$. Alternatively, we could have calculated the $\$2,661.50$ by applying the tax to the return in the future value calculation: $\$2,500 \times [1 + 0.095(1 - 0.32)]^1 = \$2,661.50$. The term in brackets, $[1 + 0.095(1 - 0.32)]$, is essentially an after-tax annual return.

The post-tax percentage return for an investment in Fund A is $\$2,661.50 = \$2,500 \times (1 + R)^1$ implies $R = 6.46\%$. This is a reduction of $9.5\% - 6.46\% = 3.04\%$. Note that it would have been simpler, but perhaps less instructive, to multiply the 9.5% return by the tax rate to obtain the performance difference: $9.5\% \times 0.32 = 3.04\%$.

The solution can also be obtained as follows:

$$\begin{aligned} & \text{Difference in post-tax return between Fund B and Fund A} \\ &= \text{Rate of return with no tax} - \text{Rate of return after 32\% tax} \\ &= 9.5\% - 9.5\% \times (1 - 0.32) \\ &= 3.04\% \end{aligned}$$

B is incorrect and is calculated as $9.5\% \times (1 - 0.32) = 6.46\%$. C is incorrect and is the reduction of future value from taxes of $\$76.00$ ($\$2,737.50 - \$2,661.50$). On a percentage basis, this is the tax rate of 32.0% [$\$76.00 / (\$2,737.50 - \$2,500)$].

- 20 A is correct. For Fund A, we can use the after-tax future value calculation from the previous question over 10 years: $\$2,500 \times [1 + 0.095(1 - 0.32)]^{10} = \$4,675.25$. For Fund B, we apply the 32% tax rate against the cumulative gain over 10 years. The future value of Fund B is $\$2,500 \times [1 + 0.095]^{10} = \$6,195.57$. The gain is $\$6,195.57 - \$2,500 = \$3,695.57$. The tax on the gain is $\$3,695.57 \times 0.32 = \$1,182.58$. The after-tax future value is $\$6,195.57 - \$1,182.58 = \$5,012.99$. The post-tax percentage return for an investment in Fund A is $\$4,675.25 = \$2,500 \times (1 + R)^{10} \rightarrow R = 6.46\%$.

The post-tax percentage return for an investment in Fund B is $\$5,012.99 = \$2,500 \times (1 + R)^{10} \rightarrow R = 7.21\%$.

Applying tax on an annual basis (Fund A) versus a tax applied on a deferred gain (Fund B) results in an annual return reduction of $7.21\% - 6.46\% = 0.75\%$.

The solution can also be obtained as follows:

Difference in post-tax return between Fund B and Fund A

= Rate of return when 32% tax is paid at end of 10 years only – Rate of return when 32% tax is paid annually

$$= \left\{ \left[(1 + 0.095)^{10} - 1 \right] \times (1 - 0.32) + 1 \right\}^{1/10} - 1 \text{ minus } \left\{ \left[1 + 0.095 \times (1 - 0.32) \right]^{10} \right\}^{1/10} - 1$$

$$= 7.21\% - 6.46\%$$

$$= 0.75\%$$

B is incorrect and is calculated by taxing the entire future value for B (not just the gain). C is incorrect and is calculated as $9.5\% - 6.46\% = 3.04\%$.

- 21** B is correct. Trading expenses are already accounted for in the gross-of-fees return, so they are not netted from the fund's 5.0 percent return. Although the employer pays the fund's investment management fees, the net-of-fees return will still be net of the investment management fees.

To calculate the net-of-fees return, we calculate the quarterly investment management fee: $1.60\%/4 = 0.40\%$. The net-of-fees return is then

$$\text{Net-of-fees return} = \frac{(1 + 5.0\%)^4}{(1 + 0.4\%)^4} - 1 = \frac{1.05}{1.004} - 1 = 4.6\%$$

A is incorrect and is calculated by including trading expenses in the calculation.

C is incorrect and is calculated by omitting the investment management fees.

- 22** B is correct. To calculate the effective annual return for the bond over the 18-month holding period, the terminal value of the investment must be determined. The terminal value consists of the price of the bond at the end of the 18 months plus the sum of the coupons reinvested at the 4 percent interest rate compounded semi-annually or 2 percent per six-month period. The coupons are \$35 each ($7\%/2 \times \$1,000$). The first coupon (received 12 months ago) is reinvested for two 6-month periods, the second coupon is reinvested for one period, and the last coupon earns no interest. Their reinvested values at the end of the 18 months are

$$\text{Terminal value of first coupon: } \$35 \times 1.02^2 = \$36.41.$$

$$\text{Terminal value of second coupon: } \$35 \times 1.02 = \$35.70.$$

$$\text{Terminal value of third coupon: } \$35.00.$$

$$\text{Sum of the value of the reinvested coupons is } \$36.41 + \$35.70 + \$35.00 = \$107.11.$$

The bond price when sold was \$950.41, so the total terminal value is $\$950.41 + \$107.11 = \$1,057.52$. The purchase price for the bond was \$1,000. The return relative over 18 months is then $\$1,057.52/\$1,000 = 1.0575$.

To calculate an effective annual return from this 18-month return relative, we would take it to the $2/3$ power (a year is two-thirds of an 18-month period) and subtract one: $1.0575^{2/3} - 1 = 3.80\%$.

A is incorrect because it omits the interest on the coupons. C is incorrect because it is the holding period return over 18 months.

- 23** C is correct. The compounded return relatives in the first three quarters are $(1 - 0.096)(1.044)(1 - 0.047) = 0.8994$. So, the account is at 89.94 percent of its original value at the start of the fourth quarter. To restore it to its full beginning value, we calculate the fourth-quarter return that will return the fund to a value of 1.0. Let the fourth-quarter return equal x :

$$0.8994(1 + x) = 1.00$$

$$x = 11.2\%$$

A is incorrect because it assumes the fourth-quarter return must offset the average of the first three quarters of returns. B is incorrect because it assumes the fourth-quarter return must offset the sum of the first three quarters of returns.

- 24** C is correct. In this problem, inflation is 4 percent a year for all five years, so a good that costs \$1.00 at Year 0 would cost $\$1.00 \times 1.04^5 = \1.216653 in Year 5. If the fund is worth \$0.50 at the end of the first year, the fund must provide the following return relative over the next four years to restore its beginning purchasing power:

$$\$1.216653/\$0.50 = 2.43331$$

Converting this to an annual return, we have $2.43331^{1/4} - 1 = 24.90\%$.

A is incorrect because it ignores inflation in all years. B is incorrect because it ignores inflation in the last four years.

- 25** A is correct. The holding period return is $21/19 - 1 = 10.53\%$. The continuously compounded rate of return is $\ln(1 + 0.1053) = 10.0\%$.

B is incorrect because it is the holding period return. C is incorrect because it is $e^{0.1053} - 1$.

- 26** B is correct. Because there are no external cash flows in this example, we can calculate the compound annual rate of growth (i.e., geometric mean of periodic returns) by using the beginning and ending prices to determine an annual rate of growth for the five years of growth: $(9.99/9.14)^{1/5} - 1 = 1.79\%$.

Alternatively, you can chain link the periodic return relatives and arrive at the same answer. The respective returns for the five years are 0.98%, -3.79%, 0.56%, 3.25%, and 8.35%. The product of $(1.0098)(1 - 0.0379)(1.0056)(1.0325)(1.0835)$ is 1.0930. Taking the product to the one-fifth power and subtracting one, we arrive at the same answer as before: $1.0930^{1/5} - 1 = 1.79\%$.

A is incorrect because it uses six years in the exponentiation (not five). C is incorrect because it is an arithmetic mean of the periodic returns.

- 27** A is correct. Statement #1 is not correct. The geometric mean will provide the best representation of the investor's terminal wealth because the periodic returns are not independent events. The gain or loss from one period affects the return in future periods. The geometric mean reflects the compounding of returns, whereas the arithmetic mean does not. Statement #2 is also not correct. The geometric mean will be lower than the arithmetic mean as long as the returns are volatile. The geometric mean will only be equal to the arithmetic mean when the volatility of returns is zero.

- 28 B is correct. Statement #3 is not correct. A performance analyst should not annualize the performance for periods of less than a year and report it as the annual performance. By doing so, the presentation assumes that the quarter's performance would continue for the rest of year, which the fund may not be able to achieve.

Statement #4 is correct. When using statistical analysis where a model is fitted using historical data, the arithmetic mean is preferred over the geometric mean. The geometric mean is affected by return volatility, whereas the arithmetic mean is not.

- 29 C is correct. When capital withdrawals are made from a fund, the number of units adjusts to maintain the identity of the NAV per unit.
- 30 B is correct. In analyzing historical portfolio returns, one must use geometric (not arithmetic) averages; thus, the formula for the real rate of return is calculated as follows (using the 10-year average real rate of return in the numerator and the 5-year average real rate of return in the denominator):

$$\left[\frac{\left(\frac{1.04}{1.03} \right)^{10}}{\left(\frac{1.005}{1.015} \right)^5} \right]^{\frac{1}{5}} - 1 = 2.97\%$$

A is incorrect because it is simply the difference in the average arithmetic rates for the period 2001–2010 minus the difference in the average arithmetic rates for the period 2006–2010:

$$(5.0\% - 3.8\%) - (0.6\% - 2.0\%) = 2.6\%$$

C is incorrect because it calculates the average real rate of return for the period 2001–2005 using arithmetic rates (*not* geometric rates) as follows:

$$\begin{aligned} & (2 \times \text{Average real rate from 2001–2010}) - \text{Average real rate from 2006–2010} \\ & = [2 \times (5.0\% - 3.8\%)] - (0.6\% - 2.0\%) = 3.8\% \end{aligned}$$

- 31 B is correct. On 31 December 2010 the accumulated value of \$1,000 invested on 1 January 2006 is equal to $\$1,000 \times (1 + 0.005)^5 = \$1,025.25$; and the 50 units that had cost \$1,000 on 1 January 2006 cost $\$1,000 \times (1 + 0.015)^5 = \$1,077.28$. Thus, $\$1,025.25$ can buy $\$1,025.25 / \$1,077.28 \times 50 = 47.6$ or 47 (rounded down to the lower whole number of units).

A is incorrect because it uses five times the difference in the arithmetic averages of the portfolio return and inflation $[5 \times (0.006 - 0.02)] = -0.07$ (i.e., a reduction factor of 7 percent to be applied to the 50 units or 3.5). The number of units purchased, rounded down, is 46 units. C is incorrect because it reverses the numerator and denominator values in the correct answer, B.

- 32 B is correct. We can calculate the approximate variance of portfolio returns as twice the difference between the arithmetic average and the geometric average of those returns.

- **For the period 2006–2010,**

$$\text{Variance} = 2 \times (0.6\% - 0.5\%) = 0.2\%$$

- **For the period 2001–2005,** the averages must be calculated from the averages shown for the periods 2006–2010 and 2001–2010:

- a For arithmetic averages, let us denote the arithmetic average of return in period i by AAR_i ; so we have

$$\frac{5 \times AAR_{2001-2005} + 5 \times AAR_{2006-2010}}{10} = AAR_{2001-2010}$$

$$\begin{aligned} \text{or } AAR_{2001-2005} &= 2 \times AAR_{2001-2010} - AAR_{2006-2010} \\ &= (2 \times 5.0\%) - 0.6\% \\ &= 9.4\% \end{aligned}$$

- b For geometric averages, let us denote the geometric average of return in period i by GAR_i ; so we have

$$(1 + GAR_{2001-2005})^5 \times (1 + GAR_{2006-2010})^5 = (1 + GAR_{2001-2010})^{10}$$

$$\begin{aligned} \text{or } (GAR_{2001-2005}) &= \left(\frac{(1 + GAR_{2001-2010})^{10}}{(1 + GAR_{2006-2010})^5} \right)^{\frac{1}{5}} - 1 \\ &= \left[\frac{1.04^{10}}{1.005^5} \right]^{\frac{1}{5}} - 1 = 7.6\% \end{aligned}$$

- c Geometric average of returns = $\left[\frac{1.04^{10}}{1.005^5} \right]^{\frac{1}{5}} - 1 = 7.6\%$

$$\text{Thus, variance} = 2 \times (9.4\% - 7.6\%) = 3.6\%$$

- **For the period 1996–2000**, the averages must be calculated from the averages shown for the periods 1996–2010 and 2001–2010:

$$\text{Arithmetic average of returns} = (3 \times 6.8\%) - (2 \times 5.0\%) = 10.4\%$$

$$\text{Geometric average of returns} = \left[\frac{1.06^{15}}{1.04^{10}} \right]^{\frac{1}{5}} - 1 = 10.1\%$$

$$\text{Thus, variance} = 2 \times (10.4\% - 10.1\%) = 0.6\%$$

- 33 A is correct. Because there are no external cash flows, we do not need to examine the holding period returns for each quarter to calculate the time-weighted return. We can simply divide the terminal value by the beginning value to obtain the following return relative:

$$\$1,118.44/\$1,000.00 = 1.11844$$

We would take this value to the $2/3$ power (one year represents two-thirds of an 18-month period) and subtract one:

$$1.11844^{2/3} - 1 = 7.75\%$$

Alternatively, the same annualized time-weighted return can be obtained using the quarterly holding period returns as follows: First, calculate each quarterly holding period return using the portfolio values. The first holding period return uses the April 2011 and January 2011 portfolio values:

$$\$1,005.12/\$1,000 - 1 = 0.0051 = 0.51\%$$

The second holding period return uses the July 2011 and April 2011 portfolio values:

$$\$1,082.11/\$1,005.12 - 1 = 0.0766 = 7.66\%$$

Note that we do not need to add the coupon in the return calculation because we are using portfolio values that are inclusive of coupons.

For the remaining holding period returns, you should calculate -2.69% , 4.50% , -1.02% , and 2.68% . Adding one to each holding period return and taking the product, we have $(1.0051)(1.0766)(1 - 0.0269)(1.0450)(1 - 0.0102)(1.0268) = 1.1183$.

If we take this to the $4/6$ power (there are four quarters in a year and six quarters total) and subtract one, we have $(1.1183^{4/6}) - 1 = 7.74\%$. Your answer would be 7.75% if you carried intermediate calculations in your calculator's internal memory.

B is incorrect because it uses only the product of the periodic return relatives. C is incorrect because it inverts the exponent on the return calculation.

- 34** C is correct. If the coupons are not reinvested, there are withdrawals (i.e., external cash flows) from the portfolio, and a holding period return must be calculated whenever an external cash flow occurs. The first semi-annual holding period return relative uses the January 2011 and July 2011 bond prices and the \$60 coupon:

$$(\$1,022.11 + 60)/\$1,000 = 1.0821$$

The second semi-annual holding period return relative uses the July 2011 and January 2012 bond prices and the \$60 coupon:

$$(\$978.92 + \$60)/\$1,022.11 = 1.0164$$

Notice that in this second holding period return relative, the denominator is an amount (\$1,022.11) that excludes the first coupon.

The third semi-annual holding period return relative uses the January 2012 and July 2012 bond prices and the \$60 coupon:

$$(\$933.90 + \$60)/\$978.92 = 1.0153$$

Chain linking the return relatives, we have $(1.0821)(1.0164)(1.0153) = 1.1167$. If we take this to the $2/3$ power (there are two semi-annual periods in a year and three total in this problem) and subtract one, we have $(1.1167^{2/3}) - 1 = 7.64\%$.

A is incorrect because it uses a one-sixth exponent in the return calculation. B is incorrect because it uses a one-fourth exponent in the return calculation.

- 35** C is correct. The money-weighted return is an IRR. The IRR satisfies the equation $\$933.90 + \$60 = \$1,000 \times (1 + R)^{3/2} - \$60 \times (1 + R)^{2/2} - \$60 \times (1 + R)^{1/2}$. The annual IRR is 7.91% . Note that the exponents in the equation are fractions of a year (e.g., the principal of \$1,000 is invested 1.5 years before the terminal period).

The IRR calculation is most easily performed by entering the beginning bond price, the ending bond price, and external cash flows (i.e., the withdrawn coupons) as semi-annual inputs in a financial calculator's Cash Flow menu. Using the BA-II Plus,

$$\text{CF}_0 = -1,000; \text{CO}_1 = 60; \text{FO}_1 = 1; \text{CO}_2 = 60; \text{FO}_2 = 1; \text{CO}_3 = 933.90 + 60; \text{CPT IRR} = 3.879986744.$$

Notice that in contrast to the time-weighted return, we do not need the interim semi-annual bond valuations here. We only need the beginning bond price, the ending bond price, and external cash flows.

The calculator's answer is a semi-annual IRR in percent (because we had used semi-annual inputs). To convert it to an annual IRR, we convert it to a decimal, add one, raise it to the second power, and subtract one:

$$(1.03879986744)^2 - 1 = 7.91\%$$

We could also calculate the modified Dietz return as an approximation to the internal rate of return. The first coupon is withdrawn with two-thirds of the 18-month period remaining, and the second coupon is withdrawn with one-third of the 18-month period remaining. Using the formula from the text,

$$R = \frac{V_1 - V_0 - \sum_{k=1}^K C_k}{V_0 + \sum_{k=1}^K W_k C_k} = \frac{\$993.90 - \$1,000 + \$60 + \$60}{\$1,000 - \frac{2}{3}(\$60) - \frac{1}{3}(\$60)} = 12.1170\%$$

This is an 18-month return. To convert it to an annual return, we add one to its decimal form, raise it to the two-thirds power, and subtract one: $1.121170^{2/3} - 1 = 7.92\%$. This approximation is very close to the IRR we had calculated.

B is incorrect because it omits the last coupon. A is incorrect because it reverses the signs of the beginning and terminal cash flows.

- 36** A is correct. Scenario A presents the situation where the money-weighted return would be the most appropriate measure of the portfolio manager's performance. For the money-weighted return to be a proper measure of manager evaluation, the manager must control the timing of investment cash flows, which this manager does because he controls the timing of investment liquidation. The money-weighted return is also useful when the investments are illiquid because it does not require periodic valuations, as does the time-weighted return. The portfolio in this scenario is invested in microcap stocks and hedge funds, which have no ready secondary market to provide asset valuations.

In Scenario B, the assets of publicly traded large-cap stocks and commodity futures are liquid and can thus be readily valued. So the time-weighted return can be used here. Furthermore, the manager has no control over disbursements and contributions, so the money-weighted return would not be appropriate. In particular, the client's divorce in the 2007 is likely to result in the portfolio return being relatively high over time because the client required a large disbursement prior to the 2008 market decline.

In Scenario C, it does not appear that the manager has had complete control over the timing of the portfolio's investments because the client had a large, one-time contribution from an inheritance. Furthermore, the client wishes to know the terminal value of an initial investment, and the time-weighted return provides the most accurate measure of the compound rate of growth of \$1 initially invested in a portfolio over a stated period.

- 37** B is correct. To calculate the time-weighted rate of return, first calculate the holding period returns using account values. The first holding period return is

$$[(\$188,000 + \$8,000)/\$237,000] - 1 = -0.1730 = -17.30\%$$

Note that we add the dividend in the return calculation because the investor receives this value and because the account values shown exclude dividends.

The second holding period return is

$$[(\$220,000 - \$40,000)/\$188,000] - 1 = -0.0426 = -4.26\%$$

The third holding period return including the dividend is

$$[\$329,000 + \$8,000)/\$220,000] - 1 = 0.5318 = 53.18\%$$

Adding one to each holding period return and taking the product, we have $(1 - 0.1730)(1 - 0.0426)(1.5318) = 1.2129$. Subtracting one, the time-weighted return is 21.29 percent. Note that the compounding is over a year and the time-weighted return desired is annual, so no exponentiation is needed.

A is incorrect because it omits the dividends in the holding period return calculations. C is incorrect because it omits the contribution in the holding period return.

- 38** C is correct. The first dividend is paid with 6/12 of the 12-month period remaining. The contribution is made with 3/12 of the 12-month period remaining. Calculating the money-weighted return as an IRR, the IRR will satisfy the equation $\$329,000 + \$8,000 = \$237,000 \times (1 + R)^{12/12} - \$8,000 \times (1 + R)^{6/12} + \$40,000 \times (1 + R)^{3/12}$. The annual IRR is 28.06 percent. Note that the equation above is in fractions of a year.

The IRR calculation is most easily performed by entering the beginning account value, the ending account value, and external cash flows (i.e., the dividends and contributions) as monthly inputs in a financial calculator's Cash Flow menu. We use months because they are the largest common denominator of time in the periodicity of the cash flows. We enter the dividends as positive values because they are paid out to the investor. Using the BA-II Plus,

$$\begin{aligned} \text{CF0} &= -237,000; \text{CO1} = 0; \text{F01} = 5; \text{CO2} = 8,000; \text{FO2} = 1; \text{CO3} = 0; \text{F03} = \\ &2; \text{CO4} = -40,000; \text{FO4} = 1; \text{CO5} = 0; \text{F05} = 2; \text{CO6} = 329,000 + 8,000; \text{CPT} \\ \text{IRR} &= 2.0824. \end{aligned}$$

Note that the zeros for the COs tell the calculator that no cash flows occur during the periods between cash flows.

The calculator's answer is a monthly IRR in percent (because we had used monthly inputs). To convert it to an annual IRR, we convert it to a decimal, add one, raise it to the 12th power, and subtract one: $1.020824^{12} - 1 = 28.06\%$. Notice that in contrast to the time-weighted return, we do not need the periodic account valuations here. We need only the beginning account value, the ending account value, and external cash flows.

The time-weighted return of 21.28 percent is smaller than the money-weighted return of 28.06 percent because the contribution was made right before a period of favorable performance (the last period's holding period return was highest at 53.18 percent). If the portfolio manager does not control the timing of external cash flows, the money-weighted return is not as useful for manager evaluation because the client's contribution and withdrawal decisions can influence the account's performance.

We could also use the modified Dietz return as an approximation to the internal rate of return here. Using Equation 25 from the text,

$$R = \frac{V_1 - V_0 - \sum_{k=1}^K C_k}{V_0 + \sum_{k=1}^K W_k C_k} = \frac{\$329,000 + \$8,000 - \$237,000 + \$8,000 - \$40,000}{\$237,000 - \frac{6}{12}(\$8,000) + \frac{3}{12}(\$40,000)} = 27.98\%$$

This approximation is close to the IRR of 28.06 percent we had calculated but not exact. The difference between the IRR and the modified Dietz return will be larger when interim cash flows are relatively large (relative to the beginning and ending values) and when the interim cash flows occur earlier. This can be seen by comparing the last terms in the formulas for the IRR and the modified Dietz return in the text.

$$\text{IRR: } V_1 = V_0(1 + R) + \sum_{k=1}^K C_k(1 + R)^{W_k}$$

$$\text{Modified Dietz return: } V_1 = V_0 \times (1 + R) + \sum_{k=1}^K C_k \times (1 + W_k R)$$

The modified Dietz return formula shown here is a rewriting of Equation 25 previously shown.

A is incorrect because it fails to recognize the appropriate chronological spacing of the cash flows and assumes the periodic valuations are consecutive in the IRR calculation. B is incorrect because it omits the dividends in the IRR calculation.

- 39** C is correct. To calculate the time-weighted return, we calculate the holding period returns, which are defined by the occurrence of external cash flows. The first holding period return from January to April is

$$[(\$111,300 - \$13,000)/\$93,200] - 1 = 0.0547 = 5.47\%$$

Note that the contribution is subtracted from the ending value in the return calculation because the fund did not earn this amount. We do not need to add the dividend in the return calculation because we are using fund values that are inclusive of the reinvested dividends.

The second holding period return from April to October is

$$[(\$49,900 + \$21,000)/\$111,300] - 1 = -0.3630 = -36.30\%$$

Note that we add the \$21,000 in the numerator because the 1 October valuation will not reflect this cash flow received by the employees.

The third holding period return from October to December is

$$(\$74,100/\$49,900) - 1 = 0.4850 = 48.50\%$$

Adding one to each holding period return and taking the product, we have $(1.0547)(1 - 0.3630)(1.4850) = 0.9977$. Subtracting one, the time-weighted return is -0.23 percent. Note that the compounding is over a year and the time-weighted return desired is annual, so no exponentiation is needed.

A is incorrect because it reverses the signs of the contribution and the withdrawal. B is incorrect because it omits the withdrawal in the holding period return.

- 40** B is correct. The contribution is made with 9 of the 12 months remaining. The withdrawal is made with 3 of the 12 months remaining. Using Equation 25 from the text, the modified Dietz return is

$$\frac{74,100 - 93,200 - 13,000 + 21,000}{93,200 + 13,000 \times \frac{9}{12} - 21,000 \times \frac{3}{12}} = -11.36\%$$

We could also calculate the money-weighted return as an IRR. The IRR will satisfy the equation $74,100 = 93,200 \times (1 + R)^{12/12} + 13,000 \times (1 + R)^{9/12} - 21,000 \times (1 + R)^{3/12}$. The annual IRR is -11.37 percent. Calculating the

money-weighted return as an IRR in a financial calculator's Cash Flow menu, we enter the beginning account value, the distribution, the contribution, and the ending account value as monthly inputs. Note that we do not need to enter the dividends because they are not outflows (they are reinvested in the account). Using the BA-II Plus,

$$\begin{aligned} \text{CF0} &= -93,200; \text{CO1} = 0; \text{F01} = 2; \text{CO2} = -13,000; \text{FO2} = 1; \text{CO3} = 0; \text{F03} = \\ &5; \text{CO4} = 21,000; \text{FO4} = 1; \text{CO5} = 0; \text{F05} = 2; \text{CO6} = 74,100; \text{CPT IRR} = \\ &-1.001 \end{aligned}$$

Note that the zeros for the COs tell the calculator that no cash flows occur during the periods between cash flows. The calculator's answer is a monthly IRR in percent (because we had used monthly inputs). To convert it to an annual IRR, we convert it to a decimal, add one, raise it to the 12th power, and subtract one: $0.9900^{12} - 1 = -11.37\%$. This is very close to the modified Dietz return of -11.36 percent we had calculated previously.

Notice that the money-weighted return of -11.4 percent is significantly lower than the time-weighted return, which is near zero at -0.22 percent. This is because the contribution was made right before a period of poor performance (the second period's holding period return was lowest at -36.30 percent) and also because a distribution was made right before a period of superior performance (the last period's holding period return was highest at 48.50 percent). Note that the timing of these external cash flows was beyond the control of the manager and was due to aspects of the pension fund's obligations to its employees. Because of this, the money-weighted return should not be used for manager evaluation here.

A is incorrect because it omits the withdrawal in the calculation. C is incorrect because it omits the contribution in the calculation.

- 41 C is correct. The number of units changes as withdrawals and contributions occur. The 22 September contribution of \$6.70 increases the number of units as follows: $\$6.70/\$5.424 = 1.235$. The fund's number of units is increased from 8.038 to 9.273.

For practice, verify the decline in the number of units from the 4 September withdrawal. The 4 September withdrawal of \$8.30 reduces the number of units by the NAV at that time: $\$8.30/\$4.230 = 1.962$. The fund's number of units is reduced from 10 to 8.038.

A is incorrect and is calculated by inverting the correct calculation. B is incorrect and is calculated by multiplying the ratio of the external cash flow to the market value by the NAV.

- 42 C is correct. The time-weighted return can be calculated using the beginning and ending NAV of the fund: $\$5.122/\$4.580 - 1 = 11.83\%$. Note that you do not need to use daily fund values for every day in September (not shown in the table) to calculate the time-weighted rate of return. You also do not need to chain link the returns from the periodic NAVs shown in the table, although this will also provide the correct answer of 11.83 percent.

A is incorrect and is calculated by chain linking the returns from the periodic NAVs and reversing the signs of the external cash flows. B is incorrect and is calculated by chain linking the returns from the periodic market values and omitting the external cash flows.

The data below complete the table in the question:

	Market Value	Units before Flow	NAV	External Cash Flow	Added Units
31 Aug	\$45.80	10.000	4.580		0.00
4 Sep	\$42.30	10.000	4.230	(\$8.30)	-1.962
13 Sep	\$37.70	8.038	4.690		0.00
22 Sep	\$43.60	8.038	5.424	\$6.70	1.235
30 Sep	\$47.50	9.273	5.122		0.00

- 43 C is correct. The IRR satisfies the equation $289 = 241 \times (1 + R) + 34 \times (1 + R)^{28/31} - 14 \times (1 + R)^{9/31}$.

The IRR is 10.46 percent. The IRR calculation is most easily performed in a financial calculator's Cash Flow menu by using daily inputs. Using the BA-II Plus,

$$CF0 = 241; CO1 = 0; F01 = 2; CO2 = 34; FO2 = 1; CO3 = 0; F03 = 18; CO4 = -14; FO4 = 1; CO5 = 0; F05 = 8; CO6 = -289; CPT IRR = 0.321485.$$

Note that the zeros for the COs tell the calculator that no cash flows occur during the periods between cash flows.

The calculator's answer is a daily IRR in percent (because we had used daily inputs). To convert it to a monthly IRR, we convert it to a decimal, add one, raise it to the 31st power, and subtract one: $1.00321485^{31} - 1 = 10.46\%$. You can practice the calculation of the IRR by determining it for Asset B. You should calculate -5.20 percent.

A is incorrect because it is the IRR if the time periods between external cash flows are ignored. B is incorrect and is calculated using a simple percentage return of the initial investment, the contribution, the withdrawal, and the terminal value.

- 44 B is correct. To calculate the modified Dietz return for December, first determine the portion of the month (W) between the contribution and month-end:

$$W = \frac{31 - 3}{31} = 0.9032$$

The portion of the month (W) between the withdrawal and month-end is

$$W = \frac{31 - 22}{31} = 0.2903$$

Next, adjust the beginning portfolio value in the numerator and ending portfolio value in the numerator by the contribution and the withdrawal as follows:

$$\text{Modified Dietz return} = \frac{\$86.00 - \$88.00 - \$12.00 + \$9.00}{\$88.00 + (\$12.00)(0.9032) - (\$9.00)(0.2903)} = -5.20\%$$

A is incorrect and is calculated by omitting the weights in the modified Dietz return calculation. C is incorrect and is calculated by reversing the signs of the contribution and the withdrawal in the modified Dietz return calculation.

You can practice the calculation of the modified Dietz return by determining it for Asset A. You should calculate 10.46 percent:

$$\text{Modified Dietz return} = \frac{\$289.00 - \$241.00 - \$34.00 + \$14.00}{\$241.00 + (\$34.00)(0.9032) - (\$14.00)(0.2903)} = 10.46\%$$

The modified Dietz return here is very close, but not equal, to the IRR calculated previously for Asset A. Using four decimal places for the percentage modified Dietz return, we would have 10.4616 percent. Using four decimal places for the IRR, we would have 10.4619 percent. The IRR and modified Dietz returns for Asset B are also very close but not exactly the same.

- 45 C is correct. The modified Dietz return for the entire portfolio will be consistent with the modified Dietz return calculated using portfolio segments when the beginning-of-period values are adjusted for external cash flows. The adjustment is implemented using values from the denominator of the modified Dietz formula.

You can confirm this consistency by calculating the modified Dietz return for the entire portfolio and comparing it with the portfolio return calculated using portfolio segments. Using the values for the entire portfolio below and the method shown in the previous answer, you should calculate a modified Dietz return for the portfolio of 6.32 percent.

	Date (t)	Asset A	Asset B	Market Value (V _t)
Market Value	30 Nov	241.00	88.00	329.00
Contribution Deposit	3 Dec	34.00	12.00	46.00
Withdrawal	22 Dec	14.00	9.00	23.00
Market Value	31 Dec	289.00	86.00	375.00

Using the segment returns where segment weights are adjusted for external cash flows, we can also calculate 6.32 percent. The beginning-of-period value adjusted using the modified Dietz formula denominator for Asset A is \$241 + (\$34 × 0.9032) – (\$14 × 0.2903) = \$267.645. For Asset B it is \$88 + (\$12 × 0.9032) – (\$9 × 0.2903) = \$96.226. The total adjusted weights are \$267.645 + \$96.226 = \$363.871.

Using the relative weighting of A and B with their respective modified Dietz returns of 10.46 percent and –5.20 percent from the previous answer results in the modified Dietz return for the portfolio of 6.32 percent: (267.645/363.871) × 10.46% + (96.226/363.871) × –5.20% = 6.32%.

- 46 B is correct. The return on a portfolio that buys and holds the assets can be calculated as a weighted average of the assets' returns using the beginning-of-period portfolio weights for the assets. Here we know the total return of Laura's account (5%). The total return of Noah's account is equal to the given price return (–1%) plus the income return (= 0.6/20 or 3% since the dividends received were held in cash). Thus, we have

$$\text{Total return} = 5\% \times \frac{80}{80 + 20} + (-1\% + 3\%) \times \frac{20}{80 + 20} = 4.4\%$$

A is incorrect. It is obtained when one ignores the 3 percent income return in the above formula. C is incorrect. It is obtained when one ignores the –1 percent price return in the above formula.

- 47 A is correct. The RESP fund return using the beginning weights of the accounts is obtained as follows:

Laura's contribution to return + Noah's contribution to return + Sophia's contribution to return (where each contribution to return = Modified Dietz return × Beginning weight) =

$$\begin{aligned} & \frac{105 - 100 - 10}{100 + 10 \times \frac{11}{12}} \times \frac{100}{140} + \frac{65 - 40 - 20}{40 + 20 \times \frac{11}{12}} \times \frac{40}{140} + \frac{31 - 0 - 30}{0 + 30 \times \frac{11}{12}} \times \frac{0}{140} \\ & = -3.27\% + 2.45\% + 0\% = -0.82\% \end{aligned}$$

Note that the modified Dietz return for the total RESP fund is obtained as follows:

$$\frac{201 - 140 - 60}{140 + 60 \times \frac{11}{12}} = 0.51\%$$

Thus, the weighted returns using beginning weights do not sum to the total portfolio return. This is not surprising given that in this particular instance, Sophia's contribution to return is zero in the RESP fund return using beginning weights simply because the beginning weight of Sophia's account is zero.

B is incorrect because in the denominator of each of the three terms in the formula above the proportion of the year applied to the new contribution is 1/12 rather than 11/12. C is incorrect because in the denominator of each of the three terms in the formula above, the new contribution is totally ignored.

- 48 A is correct. The RESP fund return using the beginning weights of the accounts adjusted for the new contributions is obtained as follows:

Laura's contribution to return + Noah's contribution to return + Sophia's contribution to return (where each contribution to return = Modified Dietz return \times Beginning weight adjusted for the new contribution) =

$$\begin{aligned} & \frac{105 - 100 - 10}{\mathbf{100 + 10 \times \left(\frac{11}{12}\right)}} \times \frac{\mathbf{100 + 10 \times \left(\frac{11}{12}\right)}}{100 + 10 \times \left(\frac{11}{12}\right) + 40 + 20 \times \frac{11}{12} + 0 + 30 \times \frac{11}{12}} \\ & + \\ & \left[\frac{65 - 40 - 20}{\mathbf{40 + 20 \times \left(\frac{11}{12}\right)}} \times \frac{\mathbf{40 + 20 \times \left(\frac{11}{12}\right)}}{100 + 10 \times \left(\frac{11}{12}\right) + 40 + 20 \times \frac{11}{12} + 0 + 30 \times \frac{11}{12}} \right] \\ & + \\ & \left[\frac{31 - 0 - 30}{\mathbf{0 + 30 \times \left(\frac{11}{12}\right)}} \times \frac{\mathbf{0 + 30 \times \left(\frac{11}{12}\right)}}{100 + 10 \times \left(\frac{11}{12}\right) + 40 + 20 \times \frac{11}{12} + 0 + 30 \times \frac{11}{12}} \right] \end{aligned}$$

The bold terms cancel, we use the common denominator, and then we are left with

$$\begin{aligned} & = \frac{(105 - 100 - 10) + (65 - 40 - 20) + (31 - 0 - 30)}{100 + 10 \times \left(\frac{11}{12}\right) + 40 + 20 \times \frac{11}{12} + 0 + 30 \times \frac{11}{12}} \end{aligned}$$

$$= \frac{(105 + 65 + 31) - (100 + 40 + 0) - (10 + 20 + 30)}{100 + 40 + 0 + (10 + 20 + 30) \times \left(\frac{11}{12}\right)}$$

$$= \frac{201 - 140 - 60}{140 + 60 \times \frac{11}{12}} = 0.51\%$$

Note that this is also the modified Dietz return for the total RESP fund. Thus, the weighted returns using beginning weights adjusted for contributions sum to the total portfolio return. The mathematical expression $\frac{201 - 140 - 60}{140 + 60 \times \frac{11}{12}}$ above is

the expression that we developed in the previous question for the modified Dietz return of the total RESP fund. Indeed, the modified Dietz return of a total portfolio is mathematically equivalent, as we have illustrated, to the sum of the weighted composite returns when the weights are the beginning weights adjusted for the contributions (and withdrawals) made.

Because the weights sum to 1 and the weighted returns sum to the total fund return, there is consistency between the segment returns (the three terms in the first mathematical expression above—that is, -2.56% , $+2.56\%$, and $+0.51\%$).

B is incorrect because in all of the expressions above, or equivalently in the simple mathematical expression $\frac{201 - 140 - 60}{140 + 60 \times \frac{11}{12}}$, the weight used for the

number of months is $1/12$ rather than $11/12$. C is incorrect because in all of the expressions above, or equivalently in the simple mathematical expres-

sion $\frac{201 - 140 - 60}{140 + 60 \times \frac{11}{12}}$, the weight applied to the contribution is subtracted rather

than added—that is, $-60 \times (11/12)$ rather than $+60 \times (11/12)$.

- 49** B is correct. The contribution to portfolio return of a segment return is the weighted return of the segment. Here, using time-weighted rate of return, there is no external cash flow involved and the weight used is the beginning weight of the segment. Hence, contribution to return is equal to

$$\text{Segment return} \times \text{Weight}$$

$$= \frac{31 - 30}{30} \times \frac{30}{125 + 60} = 0.54\%$$

Note that the contributions to return of the other two accounts for the same period would be as follows:

$$\text{Laura: } = \frac{105 - 90}{90} \times \frac{90}{185} = 8.11\%$$

$$\text{Noah: } = \frac{65 - 65}{65} \times \frac{65}{185} = 0.00\%$$

The sum of the three weights is equal to one and the sum of the contributions to returns ($0.54\% + 8.11\% + 0.00\%$) is equal to the overall fund return:

$$(201 - 185)/185 = 8.65\%$$

Thus, account returns are consistent with the RESP fund return, which is always the case with true time-weighted rates of return when segments are valued at the same dates as the total portfolio.

A is incorrect. It is equal to Sophia's account return $\times (31/201)$, where $(31/201)$ is the ending weight. C is incorrect. It is equal to Sophia's account return $\times (11/12)$.

- 50 C is correct (i.e., Statement 3 is incorrect). The money was invested for 11 months in 2010, and the rate of return for that 11-month period was 3.33 percent ($= 31/30 - 1$) and thus was below 3.5 percent. It would not be correct to annualize that rate to $1.03333^{12/11} - 1 = 3.64\%$ and make it represent the whole year 2010. Note that 3.64 percent is also the internal rate of return for the whole year 2010 [from $31 = 30 \times (1 + \text{IRR})^{11/12}$]. Also, it would not be correct to use the IRR because Ryan wants a return that is independent of the timing of the external cash flows.

A is incorrect (i.e., Statement 1 is correct). The money-weighted rate of return (IRR) of Laura's account is equal to -4.58 percent, obtained from

$$105 = 100 \times (1 + \text{IRR}) + 10 \times (1 + \text{IRR})^{11/12}$$

The modified Dietz method provides the same rate of return, -4.58 percent, calculated as follows:

$$\frac{105 - 100 - 10}{100 + 10 \times \frac{11}{12}}$$

B is incorrect (i.e., Statement 2 is correct). The time-weighted rate of return best reflects the manager's skill in investing the assets, irrespective of the timing of external cash flows. The time-weighted rate of return of Noah's account is equal to 12.5 percent, obtained as follows:

$$\frac{45}{40} \times \frac{65}{45 + 20} - 1$$